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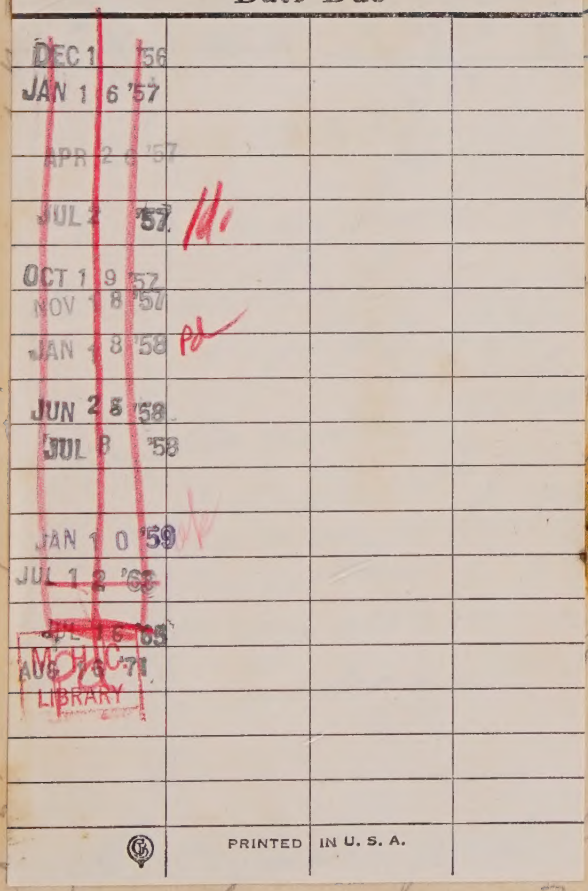

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THE TEACHING OF ARITHMETIC

A MANUAL FOR TEACHERS

A TEXTBOOK FOR TEACHER-TRAINING SCHOOLS

BY

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PREFACE

That arithmetic is often poorly organized and poorly taught is shown by the large proportion of children who fail in the subject. The long experience of the authors in training teachers and in institute work confirms them in the opinion that the teaching of the subject suffers because many teachers lack adequate instruction in its organization, principles, methods, and devices. It is not assumed that method can be substituted for scholarship, but that teachers need definite advice in the teaching of arithmetic. Teachers' reading circles, teacher-training classes in high schools, normal schools, and teachers' colleges, school supervisors, and, more especially, classroom teachers were kept in mind constantly in the preparation of this book.

The book was not designed to exploit any particular theory or method. It is a simple and clear exposition of the best methods of teaching arithmetic known today. The book should aid the teacher to make arithmetic more interesting and more vital. The motivation and the socialization of arithmetic receive emphasis, and the teacher is made familiar with the rich and suggestive literature of recent years dealing with scientific studies in arithmetic.

Part One treats of the contribution which scientific studies have made toward standardizing the subject and contains a valuable study of the uses to which arithmetic is put in everyday life. Part Two deals with certain fundamental principles that apply to arithmetic in general. Part Three explains the organization of the subject and the methods of teaching the various topics which are included in the textbooks of arithmetic in general use.

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THE TEACHING OF ARITHMETIC

PART ONE

CHAPTER I

SCIENTIFIC STUDIES IN ARITHMETIC

In education, the results of scientific studies are rapidly supplanting dogmatic statements. Previously, progress was made by the laborious process of trial and error; what seemed to be successful experience was usually the standard for judging practice. During recent years, however, discontent with opinion as the final standard for judging methods and materials has become evident in many fields of human activity. There is an ever increasing interest in experiments and investigations.

The prevailing tendency to manage commerce and manufacturing scientifically has expressed itself in education in the devising of educational scales, with their equality of units, and in the devising of standardized tests, with their analytical qualities; in extensive surveys of school achievement; in intensive studies of the effects of instruction; and in the scientific determination of the content of school subjects. In preparing his scales and in making his experiments and investigations, the specialist in education now uses scientific methods.

It is not our purpose to discuss in this chapter either the nature of these scientific methods or the reliability of the results secured, but rather the results of some of the out-

standing experiments and investigations in the field of arithmetic, and the practical uses to which these results may be put.

Influence of External Conditions. The earliest attempt to measure achievement in arithmetic on a large scale and to judge the conditions of instruction in this subject was made by J. M. Rice. In 1902 he gave tests, each consisting of eight tasks, to some 6,000 pupils in grades four to eight inclusive. He sought the answers to three questions: What results should be accomplished? How much time should be given to arithmetic? Why do some schools secure satisfactory results with a reasonable expenditure of time, while others fail to do so with a greater expenditure of time? Although the investigation lacked the refined method that has since been developed, several of the conclusions derived from Rice's study have been corroborated by the results of later investigators.

Rice found a wide range of abilities in all grades tested. This was confirmed by the later findings of C. W. Stone for both the fundamental operations and reasoning ability; by F. G. Bonser for reasoning ability; and by Clifford Woody for the fundamental operations. The wide overlapping of abilities which Rice found from grade to grade has been confirmed by the results of the investigations of Bonser, Woody, and P. J. Kruse.

It was found that superiority, mediocrity, or inferiority in arithmetic, in any school throughout a city, may generally be taken as indicative of superiority, mediocrity, or inferiority in all schools throughout that city. While the differences from pupil to pupil in the same class were much greater than the differences from class to class of the same grade, the variation of performance from school system to school system was greater than the variation of performance from one school to another in the same school system.

In seeking some one condition that would account for

the variation in achievement in arithmetic from school system to school system, it was found that no one of the following factors could be held responsible: The kind of home environment, the size of the class, the average age of pupils in the class, the amount of time given to the subject, the time of day the subject was taught, the amount of home study required, the methods of teaching, the general qualifications of teachers, the excellence of the course of study, or the suggestions to teachers by the superintendent. When all these conditions were taken together, a marked deficiency or superiority in any one of them seemed to be sufficiently compensated for by the opposite tendency in some other conditions to neutralize whatever effects it might have had.

The two factors which seemed to Rice to account for this variation in achievement were the setting up of reasonable standards of achievement, based upon the normal mental capacity of the average child, and the use of tests for measuring the progress of pupils. While neither the extreme influence ascribed to the use of tests by Rice, nor the belief that supervision functioned much more generally than any other factor in producing arithmetical abilities, was supported by the investigation made by Clifford Stone some years later, nevertheless Stone arrived at the similar conclusion that the greatest need was standards of achievement.

Early Contributions to Measurement Methods. In a more exhaustive investigation undertaken by C. W. Stone, two tests — one a twelve-minute test in the fundamental operations of arithmetic, the other a fifteen-minute test in solving arithmetic problems — were given to the 6A pupils of several typical schools in each of twenty-six school systems distributed over New England, the Middle States, and the Central West. From the data thus gathered Stone sought the answers to two questions: What is the

nature of the product of the first six years of arithmetic work? What is the relation between distinctive procedures in arithmetic work and the resulting abilities?

In the method employed in this investigation several new features were emphasized; these have since been elaborated, especially by S. A. Courtis, and incorporated in measurement systems. As the time limits set by Stone were intentionally made too short for even the best sixth-grade pupils to complete the tests in the time allotted, the tests may be considered as measures of the rate of work rather than as tests of how difficult tasks could be done. In this respect these tests were in direct contrast with the usual classroom examinations of the time, in which the pupils were generally allowed sufficient time to complete their work. A second feature of the method, which has also become incorporated in the modern tests and scales, was the preparation of a key showing exactly what answers were to be accepted. This was the factor that gave uniformity to the results and made feasible the comparison of one group of pupils with another.

A later investigation by F. J. Kelly demonstrated the striking lack of uniformity, not only in assigning school marks to pupils in arithmetic, but also in marking examination papers in arithmetic when no such key as that used by Stone was provided. Kelly found, for example, that when rating the same paper in arithmetic, experienced teachers sometimes differed by more than twenty points.

The standards used in grading papers today vary with localities, with the training and experience of the teacher, with her inclination, temperamental attitude, and personal knowledge of the pupils. One teacher grades test papers on method, another on correctness of result; one takes neatness into consideration, another, punctuation. Obviously there are many and varied factors operating to influence teachers' ratings of papers. For comparative

purposes of any kind, examination marks in arithmetic have little value unless a uniform method of scoring the papers is strictly followed. School marks in arithmetic cannot be compared from class to class unless they too are based upon uniform standards of attainment.

Still another feature contributed by Stone was the arrangement of the tasks in his tests in the order of their difficulty for children. This was accomplished by first submitting the tasks to groups of children and then arranging the tasks in the order of their difficulty on the basis of the proportion of children who did each task correctly when sufficient time was allowed for each pupil to do as much as he could.

A more recent investigation has shown that, unless the range of difficulties is fairly wide, teachers can arrange a series of tasks in the order of their difficulty little better than would happen as a matter of chance. This has an important bearing upon the measuring of the achievements of pupils in arithmetic. If a slow pupil is halted by a difficult task early in a test, he may fail to reach some of the simpler tasks toward the end that he could do if he tried.

As an answer to the first question which he proposed, Stone arrived at standards of performance for 6A pupils in each of his own tests. He also discovered that among the twenty-six school systems there were wide variations from these standards. This tendency in each grade to vary widely from school to school, which has been in evidence in practically every school survey, and to which attention is so forcibly called by the Cleveland Survey of 1915, shows the need of more adequate supervision of the work in arithmetic throughout the grades.

While the standards set up by Stone applied only to the one test in each phase of arithmetic, they gave a definite basis for measuring the achievements of this grade in these phases of arithmetic in other schools. This plan of establish-

ing standards for the several grades has been followed in all the standardized tests and educational scales. As suggested by Stone, native capacity is probably one of the reasons for the wide variation in the performances of pupils in the same class. More recently, mental-age standards have also been established in connection with several of the tests and scales in arithmetic. With the development of the more refined methods of scale-making, scores for pupils that are independent of the particular test used are now obtainable. These give measures of the abilities of the pupils somewhat as yardsticks give measures of their heights.

The answer to the second question which Stone proposed is less definite. It is evident that the various arithmetical abilities are to some extent related. As the abilities involve many common elements, this is to be expected. Just how close this relationship is has not been accurately determined since many factors operate to make the relationship seem less than it really is. As far as actual achievement was concerned, Stone found that there was a greater uniformity of results from one school system to another among the four fundamental operations than between any of the fundamental operations and the ability to solve problems.

Incentives and Practice Periods. One of the earlier experimental studies was the practice experiment of T. J. Kirby, in which he attempted to determine the conditions or factors that are responsible for gains resulting from practice. As a result of a distributed sixty-minute experiment in addition with about 730 fourth-grade pupils in which the time was measured from the middle of the first fifteen-minute test to the middle of the final fifteen-minute test, and of a similar fifty-minute experiment in short division with about 600 third and fourth-grade pupils, Kirby found that, under the conditions of a practice experiment, the pupils made a median gain of 48 per cent of their original ability in addition, and 75 per cent of their

original ability in short division. In a similar experiment in addition, in which the same materials and methods were used with adults, E. L. Thorndike arrived at practically the same results.

In the case of Kirby's experiment, the greatest gains in speed in both addition and short division were made during the many two-minute practice periods; the smaller gains were made during the fewer long practice periods; the accuracy remained practically stationary. From three to twelve weeks after making the first test-scores, tests of the power of retention were given in several of these classes, which had had only the regular work in arithmetic after the experiment. There were indications that, with the usual amount of exercise in the regular work of the school, the gains made under the conditions of the practice experiment were retained at their maximum efficiency. While a certain amount of loss in efficiency was found as the result of the summer vacation, further retention tests given the following September indicated that the pupils were still far ahead of their initial abilities and that only about thirty minutes of distributed practice were necessary to bring them back to the same degree of efficiency as was attained at the end of the original sixty minutes of practice in the original experiment.

Kirby explained the large amount of gain made during the practice experiment as mainly due to the setting up of an attitude which resisted distraction and to the greater incentive to achievement as the result of the pupils being stimulated to excel their own previous records and of being kept informed of their scores and of their progress.

Sex Differences in Arithmetical Abilities. F. G. Bonser's study of the reasoning abilities of school children revealed some striking sex differences in the abilities to solve arithmetic problems. The boys were found to be con-

siderably superior to the girls in this respect. The same tendency has been evidenced by the distributions of scores of elementary school pupils as well as of adults in the arithmetic problem test of the well-known *Army Alpha* mental test. Results of recent surveys made in the public schools of St. Paul, Minnesota, indicate that the boys made higher scores than the girls in the *Buckingham Scales for Arithmetic Problems* even though the girls surpassed the boys in their ability to handle the fundamental operations in arithmetic, as revealed by the revised *Woody Arithmetic Scales*.

The Development of Measuring Instruments. A different type of scientific study in arithmetic is represented by the work of Clifford Woody in constructing a series of tests or scales for measuring achievement in the four fundamental operations of arithmetic. The earlier work of Courtis resulted in the development of tests in which the tasks set for the pupil were of equal difficulty. These tests were designed for measuring speed and accuracy rather than the difficulty of the operations that the pupil could perform successfully. In devising his scales, Woody assumed that arithmetical abilities are distributed in practically the same manner as physical traits; that is, that very few pupils possess very little arithmetical ability, but that more and more pupils possess greater and greater amounts of ability in any grade. The average amount of ability for that grade is possessed by the largest proportion of pupils. When the average is passed, however, the larger amounts of ability are possessed by smaller and smaller proportions of the group, and the very large amounts of ability are possessed by but a very small number of pupils. The distributions of scores which Woody found in working out his study tend to substantiate the assumption that he made.

In the derivation of the Woody scales, the important

fact came to light that it is in the difficult tasks that the larger differences in achievement are apparent from grade to grade. Likewise, the smallest differences between consecutive grades are found when the easier tasks only are used. Other tests, less scientific in their construction than the Woody scales, as well as revisions of the original Woody scales, have been developed for the measurement of achievement in the fundamental operations. Somewhat similar scales, as well as standardized tests, have been worked out by Buckingham, Otis, and Monroe for measuring the ability to solve arithmetic problems.

Analysis of Errors in Arithmetic. G. D. Counts used the *Cleveland Survey Tests in Arithmetic* to make an intensive study of the relations of correct results to the number of examples attempted in the case of 1,376 children in grades three to eight inclusive, and also of the per cent of accuracy in the case of 2,400 pupils in the same grades. In this study he found that one of the effects of school training in arithmetic is to increase individual differences rather than to make the achievements of pupils more alike. While both the bright and dull pupils make progress in acquiring arithmetical abilities, the bright pupils make more absolute progress.

In the fundamental operations involving whole numbers, in cases where the pupils are all familiar with the operations, there is a greater tendency for pupils to vary from one another in the number of examples correctly worked than in the number attempted, especially when the examples involve long processes. In the case of fractions, on the other hand, when the tasks are simple but the operations are not familiar to all, there is a greater degree of variation in the number of examples attempted than in the number right. It is also evident that the number of examples attempted increases more rapidly than the number of examples which are correct, in the case of fractions up to

the end of the seventh grade. Up to this point there is a tendency among pupils to substitute easy methods, such as adding both the numerators and denominators of fractions, for correct methods. The decrease in the proportion of the number of examples attempted to the number of correct results in the eighth grade is probably due to the elimination of faulty methods as a result of greater familiarity with the processes involved in fractions.

The greatest amount of inaccuracy is found in the processes involving long mental strain without a frequent change in the type of the mental operation required. These processes include fractions, short division, addition of long columns, addition involving carrying, and the multiplication of four-place numbers by two-place numbers. Likewise, the greatest increase in accuracy from grade to grade is found in the more complex types of examples. Although there is a uniform increase from grade to grade in the kinds of tasks correctly done, the increase in the per cent of accuracy from grade to grade is very rapid at first. Like the increase in the rate of learning, it then slows up, which indicates that accuracy is probably rather the result of proper practice. Increase in speed probably depends rather more upon the maturing of the pupil.

A detailed study of the errors made in the same test by the fifth and eighth-grade pupils of two cities also indicated several important facts. In simple addition combinations such as $7+8$, most of the errors were made on combinations whose sums are more than 10. The persistency of error in certain combinations, and the rather strong persistence of these wrong associations from the fifth to the eighth grade in the same city, indicates that such associations are probably established in a group through peculiar methods of instruction, or through some other experience common to the individuals making up the group. Gary C. Myers has recently called attention to the persistency of the same

errors unless a definite and continuous procedure is followed to eliminate them.

In summarizing the kinds of errors found in a recent survey, W. J. Osburn found that, in addition, 25 per cent of the errors were errors in combinations, and 8 per cent of the errors were due to mistakes in carrying; in subtraction, 40 per cent of the errors were in bridging, and 20 per cent of the errors involved zero difficulties; in multiplication, 5 per cent of the errors involved zero combinations, and 50 per cent were due to inaccuracies with the fundamental combinations; in division, 13 per cent of the errors were due to zero difficulties, and 43 per cent were errors in the combinations. Wide variation in the amounts of practice given to the different combinations, with the smaller amounts given to the difficult combinations, especially to the combinations involving zero, are probably responsible for these conditions.

A given process may vary in difficulty from one operation to another. For example, bridging the tens was found to be a more difficult operation in subtraction than in addition. The comprehension of the zero concept seems to offer considerable difficulty in both subtraction and multiplication, with a still different difficulty in division. The small amount of attention given to the zero combinations may explain the numerous errors. The fact that multiplying a quantity by zero offers more difficulty than multiplying zero by a quantity, and the additional fact that difficulties with zero in the simple combinations seem unrelated to difficulties with zero in complex situations, indicate that there is probably comparatively little transfer from one application to another.

F. B. Knight has shown that when all the elements involved in an operation are familiar, and when the process is thoroughly learned in connection with one kind of material, as in the addition of fractions having even numbers

for denominators, there is a large amount of transfer of ability when the same process and other kinds of familiar material are employed, as in the adding of fractions having odd numbers for denominators. When the common element is a more general one, however, such as neatness in written work in one class affecting the neatness of written work in another class, practically no transfer was found by Ruediger unless the common element was emphasized and developed into an ideal.

The types of persistent error found in handling fractions, such as adding numerators and denominators, suggest that the teaching of fractions results in following a method or in a mechanical procedure, rather than in a clear comprehension of the processes involved.

Another type of error to which attention is called in Counts' investigation is the tendency of strong associations, like the additive associations of $3+3=6$ and $7+7=14$, to assert themselves in the midst of another operation, as in a set of multiplication examples when the pupils are expected to be multiplying.

Counts' comparison of different age and promotion groups of pupils in the same school system on the basis of their achievements in the *Cleveland Survey Tests in Arithmetic* also furnishes some illuminating information. Although the number of examples attempted was approximately the same for the different age-groups in any grade, the number of correct results and the per cent of accuracy increased from the older age-group to the younger age-group in each grade, with the greatest amount of increase in the more complex tasks. In the case of promotion-groups, the pupils who had skipped grades were superior to the average pupils, while these pupils were superior to those who had repeated grades.

Teachers' Marks in Arithmetic. The fact that the difference between pupils who were repeating and those

who were skipping grades was not as great as the difference between the youngest and oldest age-groups suggests that pupils are sometimes held back from extra promotions because they are considered too young rather than because they lack the ability to do the work of a higher grade. The low scores of pupils repeating grades in comparison with the scores of the pupils making regular progress call attention to the questionableness of the effect of such repetitions. This is especially pertinent in the case of arithmetic.

The results of the Cleveland survey of 1915 indicated that, with the exception of grammar in the eighth grade, arithmetic furnished the highest proportion of failing marks in the grades above the second in the Cleveland schools at that time. Even though a large part of these failures may have been due to inability to read the problems in arithmetic, and were, therefore, really failures in a certain type of silent reading, arithmetic received the responsibility for the failures without furnishing any clues for proper preventive measures.

The relation of teachers' marks to achievements in arithmetic has recently been investigated. F. J. Kelly found some years ago that when the pupils in the upper grades were grouped into three groups equal in number and according to their marks in arithmetic, only a few more than 50 per cent remained in the same group during the succeeding year. More recently it has been found that different teachers in the same school system give pupils of equal ability in the arithmetic tests marks that differ on the average as widely as *B* and *D*. This is due in part to differences in the methods and standards used in scoring test papers, and in part to differences in subjective estimates as to what constitutes a proper standard of achievement in arithmetic in any grade.

Practice Exercises and Textbooks. H. V. Holloway attempted to determine the relative difficulty of the elemen-

cary number combinations in addition and multiplication. He listed the combinations in order of their relative difficulty from four points of view.

a) The difficulty as shown in learning the combinations.
b) The difficulty as shown by errors made in the combinations.
c) The difficulty as shown by the combinations which are forgotten.
d) The difficulty as shown by the time required to write the combinations.

An investigation of three of the best number primers led Holloway to conclude that there is no necessary relationship between the number of times a combination is presented and the relative difficulty of the combination.

Among the conclusions which Holloway reached as a result of his investigation, the following should be especially noted: Both the direct and the reverse forms should be taught in addition and in multiplication. For example, it is necessary to teach both 7+8 and 8+7. Girls seem to learn both the addition and the multiplication combinations more readily than boys, and make fewer mistakes. In the addition combinations, relatively few errors occur in the double numbers such as 5+5. The group of pupils showing least skill in manipulation is the same as the group showing the greatest number of errors.

W. J. Osburn's careful analysis of the frequency of the number combinations in two of the leading practice exercises and also in the first book of a widely used series of arithmetic textbooks, further illustrates the extent to which scientific studies in arithmetic are being carried. It is evident from Osburn's study that the greater amount of practice is given to those combinations which are easy, and the smaller amount to those which are difficult. For example, the addition and subtraction combinations involv-

ing bridging receive less practice than do the easier ones, while the combinations involving zero are seriously neglected. Wide differences occur in the number of repetitions provided for the different combinations; some 500 of the 1,470 combinations listed by Osburn do not occur at all in the practice exercises examined, while others occur as often as 92 times. Although less than two-thirds of the fundamental number combinations occur in either of these practice exercises, the provision for practice is still less adequate in the case of the textbook. On the average, the more difficult combinations receive only from one-third to two-fifths as many repetitions as the simpler combinations; while the range of repetitions varies from zero for about 180 of the combinations to 335 for others.

E. L. Thorndike has shown that reputable texts in arithmetic furnish much less practice in the use of the higher digit combinations such as $8+9$, $9-7$, 8×9, and $72\div9$, than that furnished for such combinations as $1+2$, $2-1$, 2×2, $4\div2$. Furthermore, the amount of practice in the use of the higher digits with two-place numbers in such combinations as $18+9$, $38+9$, and the like, was also found to be relatively small in comparison with that furnished in the use of smaller digits with two-place numbers containing smaller digits, as $12+2$, $13+1$, and the like. As summarized by Thorndike[1] "even gifted authors make instruments for instruction in arithmetic which contain much less practice on certain elementary facts than teachers suppose; and which contain relatively much more practice on the more easily learned facts than on those which are harder to learn."

Determining the Content of the Curriculum. W. S. Monroe made an analysis of the kinds of problems contained in four different arithmetic textbooks. The analysis revealed a condition with respect to problems similar to

[1] E. L. Thorndike, *New Methods in Arithmetic*. Rand, McNally and Company.

what Osburn found in the case of the number combinations. Of the 1,023 different types of problems found in the four textbooks, one-half occurred in only one book. Either some types are useless, and consequently have no place in a textbook, or many types of problems that ought to be in the textbooks are omitted. The variation in the number of problems found illustrating each type was even more striking. The number of repetitions afforded each type of problem varied from 1 to 108. One repetition is probably as nearly worthless as 108 are needless.

G. M. Wilson's investigation for determining what kinds of arithmetic problems are used in social and business activities affords the same kind of basis for building up the content of the arithmetic curriculum as the pioneer work of L. P. Ayres furnished for evolving a socially useful list of spelling words for children.

From a list of problems collected from more than 4,000 persons through a tabulation by their children of all the arithmetic problems which these parents had to solve for a period of two weeks, it became evident that the majority of adults ordinarily use arithmetic only when something is bought or sold or when some measurement is necessary.

When the problems had been classified under some ninety topics which included amusements, clothing, grain, lumber, music, tests, and the like, an analysis of four of the hundred fifty-five occupations listed — housekeeping, farming, mechanics, merchandising — indicated the commonest of the topics for various occupations, since there were very few topics in the entire list which did not appear under at least one of these four occupations. An analysis of the problems from the standpoint of the processes involved showed that six processes — addition with 20.1 per cent, subtraction with 12.9 per cent, multiplication with 31.8 per cent, division with 11.1 per cent, fractions with 9.0 per cent, and simple accounts with 5.5 per cent — include more than 90

per cent of all the processes used. If simple denominate numbers and percentage were to be added to these six processes, little would be left for all the others. Again the relative frequency of the use of these processes in the four occupations intensively treated follows very closely the relative frequency of their use in all occupations. The chief characteristic of the problems is their simplicity. Of the 4,416 addition problems, only 72 involved more than four-place numbers; in subtraction very few problems involved more than four places; of the 6,974 multiplication problems, there were only 53 four-place multiplicands and no five-place. The 2,437 division problems included only 102 problems with four-place divisors, and only one with a five-place divisor. Fractions occurred only about once in each eight problems, and of all the fractions used the fraction $\frac{1}{2}$ made up nearly two-thirds of the total. From this study it is apparent that several of the traditional processes might well be replaced by more adequate attention to speed and accuracy in the simpler processes.

Woody's study of 4,661 bills of sale revealed similar conditions. Of 985 addition processes only 26 involved five-place numbers; of 3,314 subtraction processes there were only 15 five-place numbers; only 8 out of 4,208 multiplication processes had either a four-place multiplier or multiplicand; while in only 52 out of 656 division processes did either the dividend or the divisor have as many as four digits. It was found that decimals and percentage were used largely in connection with United States money. From these studies it is apparent that the schools are emphasizing many unessential processes in arithmetic.

A more subtle phase of the scientific study of arithmetic is represented by much of the work of Thorndike. He has presented a series of situations or tasks that, from the analytic point of view, correspond to the arithmetical situations that are to be met with in actual life and that

correspond in their presentation to the way in which they are actually encountered, or in a way that best conforms to the nature of the learning process, where this is the more important element. For example, the artificial language of the usual arithmetic problems is replaced by a simpler form of expression that makes the arithmetical process the significant feature of the problem. Again, concrete and varied situations are provided for in the formation of arithmetical habits that lead to their functioning in all sorts of actual situations.

Thorndike has made great progress in the reduction of such vague functions as addition, subtraction, fractions, and the like into their constituent elements, and in analyzing the definite situation-response connections involved in these processes. This is well illustrated in his psychological analysis of ordinary column addition, which is usually thought of by teachers as a simple process of using the number combinations up to 9+9, automatically increasing the size of the second place number where this needs to be done, and carrying at the end. In place of this apparently simple procedure, he isolates the following distinct factors in the process,[1] each one of which is psychologically distinct and requires distinct educational treatment.

A. Learning to keep one's place in the column as one adds.
B. Learning to keep in mind the result of each addition until the next number is added.
C. Learning to add a sum to a thought-of number.
D. Learning to neglect an empty space in the column.
E. Learning to neglect 0's in the column.
F. Learning the application of the combinations to higher decades may, for the superior pupils, involve as much time and labor as learning all the

[1] E. L. Thorndike, *The Psychology of Arithmetic*, p. 52. The Macmillan Company, 1922.

original addition tables. Even for the most gifted child the formation of the connection 8 and 7 = 15 probably never quite insures the presence of the connections 38 and 7 = 45 and 18 and 7 = 25.

G. Learning to write the figure signifying units rather than the total sum of a column. In particular learning to write 0 in the places when the sum of the column is 10, 20, etc. Learning to carry also involves in itself two distinct processes, by whatever way it is taught.

Thorndike's analysis of what kinds of number habits should be formed, of how well the different kinds of habits should be formed, of the order in which they should be formed, and of how they should be combined into more complex abilities to deal intelligently with complicated arithmetical situations, as well as of the learning conditions under which they should be formed, furnishes an exceptionally striking illustration of the significance of the application of well-known psychological principles to definite learning and teaching situations in a school subject.

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See also *Suggested Readings* on page 34.

CHAPTER II

TESTING IN ARITHMETIC

In arithmetic, as in the other subjects of the elementary school, the testing process has two distinct functions. One is the measurement of the ability to perform arithmetical operations or the ability to solve arithmetical problems. The other function is concerned with finding out the specific difficulties which each pupil is encountering in his work. The second function is analytical and diagnostic in character. Some tests are designed primarily as survey tests for measuring the pupils' abilities; others are primarily diagnostic. Most of the tests, however, embody both functions to some extent.

Educational Scales in Arithmetic. As examples of the survey type of tests for determining the difficulty of the tasks which the pupils can do, the *Woody Scales in the Fundamental Operations of Arithmetic* and the *Buckingham Scales for Arithmetic Problems* are excellent illustrations.

When first devised, the Woody scales gave class-scores in terms of scale values. A revision of these tests gives also the scores for individual pupils in terms of scale values. This feature is a very desirable one in the survey tests, as it makes possible comparisons of achievements from class to class and comparisons of gains from grade to grade in a much more definite form than is possible with the ordinary standardized test, in which the scores are expressed in terms of the number of tasks correctly done.

The *Buckingham Scales for Arithmetic Problems* yield scale-scores which render definite comparisons of achievements or of gains possible in somewhat the same way that

comparisons of gains in the heights or weights of pupils may be made. The devising of these arithmetic tests, or "educational scales," as they have come to be called, has involved a great amount of painstaking and patient work, for their construction is a complicated process. Their use, however, is quite simple.

When giving the tests the teacher passes out the printed papers containing the tasks and starts the pupils to work under the usual quiet conditions of classroom instruction. As rapidly as the pupils complete their work the test papers should be handed in. At the end of a certain amount of time, specified in the directions which accompany the tests, all the papers should be collected. The time limit, however, is ample for each pupil to do as many of the tasks as he is able to do. With each set of test papers and directions a key containing the answers that should be accepted as correct is also supplied. All other answers should be marked wrong, otherwise no valid comparison can be made with the test standards.

The scoring of all the papers of a class is the work of but a few minutes. There is no feeling of uncertainty in marking the papers, and there are no differences in scores when the papers are marked by different teachers, for any answers unlike those in the key are marked wrong. The finding of the pupil's score is likewise a simple process. By noting the number of wrong answers in the hardest group of tasks on the pupil's papers, and consulting a simple table, the pupil's score is read off. If errors have been made below the hardest group of tasks, these errors are subtracted from the pupil's score.

When a given pupil's score is thus found it has a definite, tangible meaning. It indicates the difficulty of the tasks that the pupil can do with a correctness of 50 per cent, the difficulty of such tasks as he can attempt with an expectation of succeeding in about fifty out of each one hundred

of like difficulty attempted. The teacher can look at the test paper and see exactly what tasks of this difficulty are like.

Recently mental-age standards have been worked out for both the revision of the *Woody Scales in the Fundamental Operations of Arithmetic* and for the *Buckingham Scales for Arithmetic Problems.* As classes differ in mental-age from one another, the mental-age standards are even more valuable for purposes of comparison than are the grade-standards or norms. The teacher can determine not only how the class compares with other classes of the same grade, but how the score of each pupil as well as the score of the class compares with the corresponding mental-age norms.

When the tests have been scored, the scores may readily be put into a distribution table to show the achievement of the class as a whole. In comparing distributions of scores in the revised Woody arithmetic scales made by four sixth-grade classes at the close of the year, for example, it is apparent at once that the classes are not all doing the same quality of work in arithmetic. On the chart on page 24, the numbers at the left indicate the scale-scores made by the pupils. The numbers under each school indicate the number of pupils in the sixth-grade class in that school making each particular score.

The chart shows that the scores in School A range from 46 to 76, and those in School D range from 66 to 96. Thus, while the scores in each school spread out over 30 points or approximately five school grades, the extreme scores at each end of these two distributions differ by 20 points, or by more than three grades. The class-scores, which are found by counting just halfway through the distribution from either end, and which are indicated in the chart by solid horizontal lines, differ by 13 points or by more than two grades.

TABLE I

DISTRIBUTION OF SCORES MADE BY FOUR SIXTH-GRADE CLASSES AT THE END OF THE YEAR

Scale-score	School A	School B	School C	School D	
46	1				
47					
48					
49	1				
50	1				
51					
52					
53	1				
54					
55	1	1			
56		1			
57					
58					
59	1				
60	1	1			
61	1				
62	2				
63	1	1			
64	2				
65	4	3			
66	1			1	
67	6	3			
68	3	1		2	5th grade standard
69	1	2			
70		3	1	2	
71	6	1	1	1	
72	1	1	4		
73	1	1			
74	3				6th grade standard
75		2	1	5	
76	1	3		3	
77		2	1	3	
78		4	2	4	
79		3		3	
80		1	2	2	7th grade standard
81		1	2	8	
82		1		2	
83		2		5	
84				3	
85			5	2	
86		1			8th grade standard
87		1			
88			1		
89					
90					
91				1	
92			1	1	
93			1		
94					
95			1	3	
96			1	1	
Number	40	40	24	52	
Median Mental-age	12 yr. 1 mo.	12 yr. 5 mo.	12 yr. 6 mo.	12 yr. 11 mo.	

It is significant that in School A the scores below the class-score are distributed much more widely than those above the class-score. Why should there be such a wide difference between the class-scores of these two classes? Sometimes such differences are due to differences in the capacities of the pupils. The class mental-ages in the case of these two classes differ by only ten months. As compared with the class mental-age scores, which are indicated by the dotted horizontal lines, the pupils in School A are as far below their class mental-age norm — about two-thirds of a year — as those in School D are above their class mental-age norm. Over a year of the difference in their class-scores, then, is evidently due to differences in instructional conditions and it is these conditions for which teachers are responsible. The difference may have been due to the previous preparation of the pupils, but it is certainly not due to the size of the class, as the class in School D is the larger.

The classes in schools B and C show nearly as wide a range of scores as those in schools A and D. Although their class mental-ages are nearly the same, their class-scores in arithmetic differ by almost a whole grade. Such conditions as those existing in School A call for rigorous diagnostic testing and wisely planned remedial work. Likewise, the pupils with very low scores in each class, especially those in the class of School A, are in need of diagnostic and remedial attention.

As examples of what may be expected of pupils of average mental-age, doing an average quality of work, at the end of grades six, seven, and eight, the following tasks are given. The list contains tasks of such difficulty in each case that the average pupil in each of the grades indicated should be able to succeed in doing about 50 per cent of the tasks attempted.

GRADE 6	GRADE 7	GRADE 8
In Addition:		
2 ft. 6 in. 3 ft. 5 in. 4 ft. 9 in.	$25.091+100.4+25$ $+98.28+19.3614=$	$8\frac{5}{8}$ $3\frac{4}{5}$ $9\frac{2}{3}$ $7\frac{3}{4}$
In Subtraction:		
$27\frac{1}{4}$ $18\frac{2}{5}$	$\frac{5}{12}-\frac{2}{10}=$	
In Multiplication:		
$\frac{1}{2}\frac{2}{5}\times\frac{1}{3}\frac{5}{2}=$	3 ft. 5 in. 5	$.0963\frac{1}{5}$ .084
In Division:		
$.003\overline{).0936}$	$24.8\div1.6=$	$531\overline{)37722}$

In connection with both the revised series of the Woody arithmetic scales and the *Buckingham Scales for Arithmetic Problems*, the mental-age norms that have been worked out enable the teacher to determine the pupil's achievement quotient or accomplishment ratio in arithmetic. From a pupil's score in any one of the arithmetic scales, his arithmetic-age may be read off from a sheet of mental-age norms. By dividing this arithmetic-age by the pupil's mental-age, as determined by a mental examination, the teacher has a measure that tells her the per cent which this pupil's ability is of the ability of pupils in general of his own mental-age. Where the mental-ages of the pupils are known, the unusually high or unusually low scores of individual pupils may be found simply to correspond with the mental-age norms rather than to indicate a high or low quality of work, signifying that the pupils are mentally misplaced in the grade. High or low class-scores may likewise be found to correspond with the class mental-age.

Recently the highest standing class in an entire sixth grade of a large city school system in the two arithmetic scales used was found to be doing the poorest quality of work when their class-scores were compared with their class mental-age norms. The lowest class in achievement in arithmetic in the same sixth grade ranked well up toward the middle of the group of classes in their quality of work because the class mental-age was very low.

When it is possible to give several of the tests at the beginning of the year, so that a reasonably accurate measure of the pupil's ability is obtained, a reasonable goal may be set for each pupil to work toward. Setting the same goal for all pupils to attain, regardless of their previous attainments in arithmetic or of their capacities to achieve, is a very questionable procedure, as it is likely to result in discouragement and loss of interest on the part of some pupils or lead to the neglect of other subjects, while for other pupils the goal has already been attained or is too easily reached to act as a stimulus to the best quality of work.

Standardized Tests in Arithmetic. Among the standardized tests in arithmetic that may be used for survey purposes, the following should be mentioned: The earlier Woody scales; the Woody-McCall scales for measuring abilities in the fundamental operations; the *Peet-Dearborn Progress Tests* in the fundamental operations; the *Monroe General Survey Tests in Arithmetic*; the *Otis Reasoning Test*; and the *Monroe Standardized Reasoning Tests*. The scores on these tests indicate the number of the specific tasks that the pupil has done correctly.

In this respect the tests are distinctly different from the educational scale-scores, which indicate the difficulty of tasks that can be done with a certain degree of correctness. In the scales, the score is independent of the particular tasks used in the tests. It is thus a more definite measure of the pupil's ability in the subject. It indicates the kind

of tasks that can be done with a correctness of 50 per cent. If simpler and simpler tasks were given to the student, he would succeed in a larger and larger proportion of them. Were more and more difficult tasks set for him to do, he would be successful in a smaller and smaller proportion. The pupil's scale-score also suggests to the teacher the difficulty of the tasks that may be used with pupils of such widely differing amounts of abilities as are likely to be found in any class.

The *Courtis Standard Research Tests in Arithmetic* are of an entirely different nature. In these each test consists of a set of similar tasks of equal difficulty in each of the four fundamental operations. The pupil is given a definite amount of time — four minutes for each set — in which to do as many of the tasks as he can. A key giving the correct answers enables the teacher to score the tests with exactness. With these tests, both the pupil's accuracy of performance and rate of performance are measured. Class record sheets, on which the scores of the pupils are readily entered, enable the teacher to determine an accuracy score and a rate-of-performance score for the class. The standards or norms for both accuracy and rate, worked out on the basis of a large number of cases, give the teacher a basis for an evaluation of the achievement of her class in these respects.

The *Standard Supervisory Tests in Arithmetic*, more recently devised by Courtis, serve a still different survey purpose. *Test A*, which is designed for grades 4B, 4A, and 5B, may be given early in the school year, to determine which pupils need further practice on the combinations in the four fundamental operations in which carrying is not involved. At the end of a given period the test will show what has been achieved. In the same way, *Test B*, which measures the skill of the pupils in the four fundamental

operations when carrying is involved, is for use in grades 5A to 8A inclusive.

Like the *Courtis Standard Research Tests*, these tests have definite time allowances which vary from grade to grade. The class tabulation sheet accompanying the tests enables the teacher to determine at a glance the pupils who are of a standard ability for the grade according to the Courtis standards, those for whom the regular work will furnish sufficient drill, those in need of thorough drill, those in need of special attention and extra drill, and those for whom some special adjustment of work must be made.

The scales and tests so far discussed are designed primarily to measure the pupil's achievement status in some respect. Important as this is, it is equally important to discover the conditions which account for the achievement status of each of the pupils. The survey type of tests indicates where there is a need for detailed diagnostic testing. In a very general sense, all the survey tests are diagnostic in showing the achievement status of the pupils.

For locating a pupil's specific difficulties, a much more detailed analysis of his work must be made than is involved in the finding of the test-scores. In the case of such tests as the Woody arithmetic scales or the Woody-McCall tests in the mixed fundamentals, further diagnostic use may be made of the results by finding out just what kinds of tasks are missed by the pupils. A good illustration of a diagnostic test is to be found in the *Cleveland Survey Tests*.

In this test there are some fifteen groups of tasks with from eight to sixty-five tasks in a group. The tasks in each group are of the same kind rather than equal in difficulty. The first group, for example, consists of a series of sixty-five one-place addition tasks of two numbers each. The second group contains a similar set of fifty simple tasks in subtraction. Later in the test there is a second group in addition where single-place column addition is involved.

Later still in the test is found column addition involving four-place numbers. The whole test is built up on a cycle arrangement with some new factor or factors introduced into each higher cycle. An illustration will make this feature clear.

SAMPLE OF THE FIRST TEST IN ADDITION

5	0	6	2	4	5	1	6	3	7	9	0	4
7	4	3	1	8	9	0	2	3	4	8	6	5

SAMPLE OF THE SECOND TEST IN ADDITION

3	1	4
7	4	0
2	1	8
9	8	5
6	4	5

SAMPLE OF THE THIRD TEST IN ADDITION

2	3
1	1
6	7
6	6
1	9
8	6
6	3
9	4
7	5
5	4
2	2
2	9
4	9

SAMPLE OF THE FOURTH TEST IN ADDITION

8243	7337
6429	2674
9298	9669
7404	5114
2575	8154

While the test has a definite time limit, and provision is made for determining the accuracy as well as the rate of performance, this is done for each particular test rather than for the test as a whole. As a consequence, it is clear

from the pupil's score just what types of problems are giving him special difficulty. Furthermore, the teacher can analyze each test paper and thus determine what specific operations are causing the pupil his difficulty. In scoring the test paper himself the pupil also becomes aware of his specific failures and will thus accept the teacher's suggestions more purposively.

The *Monroe Diagnostic Test in Arithmetic* is built up on a similar plan. This test, however, is broken up into four parts for convenience in giving the test. This feature makes it feasible to test only those processes and operations with which the group is immediately concerned.

One of the most comprehensive series of diagnostic tests available are the *Wisconsin Inventory Tests* devised by W. J. Osburn. Each of the eight tests making up the series thoroughly covers some phase of the fundamental operations in arithmetic. Test I covers the 100 combinations in first-decade addition; Test II, the 100 combinations in first-decade subtraction; Test III, the 100 combinations of simple multiplication; Test IV, the most difficult of the combinations needed in short division; Test V, the combinations, other than those in Test I, that are most useful in column addition; Test VI, the addition combinations needed for accurate carrying in multiplication; Test VII, all the combinations that give zero in the quotient; Test VIII, all the major difficulties in long division.

As the content of these tests is based upon a thorough analysis of the number combinations and processes involved in the fundamental operations of arithmetic, their use affords the teacher an unusually comprehensive instrument for locating the pupils' difficulties. Although it takes some time to give such a series of tests, their use, by making it possible for the efforts of each pupil to be concentrated on those elements which are most in need of practice, eventually results in the saving of a large amount of time as well as

in the elimination of individual difficulties when proper remedial work follows the use of the tests.

Informal Diagnostic Tests in Arithmetic. Supplementing the survey and diagnostic tests which have been described, the classroom teacher needs to make frequent use of informal diagnostic tests of her own devising, just as she gives many informal spelling tests. These tests may be made up in a manner similar to that of the diagnostic tests already described. The teacher can make up series of tasks covering the material which she has recently gone over in class. Such tests may be used in the same way as the printed tests. While there will be no standards available, the tests may be used in such a way that each of the pupils will become aware of his own difficulties and will work to eliminate them. It is this aspect of the testing movement that is now in most need of emphasis.

Pupils should be taught to feel that tests are not given to measure their ability for the purpose of assigning marks, but rather, that they are devices for the pupil to use to see how much progress he is making and also to enable him to discover and to overcome the difficulties which stand in the way of his progress. Whether of the formal or informal type, tests are, fundamentally, teaching devices; they must come to be looked upon as such by pupils and teachers alike.

As illustrations of tests prepared by the teacher, a simple test such as the following might be used to determine whether the pupils are having difficulties in the use of the zeros, where no carrying is involved, and to locate such difficulties definitely.

Add:

				4	0	5					
3	0	5	0	0	2	3	14	106	907	305
0	6	0	7	3	4	0	10	92	72	204

Subtract:

4	12	306	107	908
0	10	203	103	606

Multiply:

14	30	102	304	303	200	40	70
20	3	4	20	102	8	30	90

Divide:

$4\overline{)80}$ $8\overline{)408}$ $20\overline{)4080}$

A test like the following would help the teacher and pupils to locate difficulties in column addition.

					9	8					
			4		3	2					12
		3	2	7	4	6					4
3	9	2	5	8	8	4	12	24	78	20	5
7	8	6	3	6	7	3	7	9	16	18	3

		15		79					53
15	12	4	88	61	206	135	214	796	325
4	10	10	70	78	14	207	364	859	8
13	13	3	96	62	23	130	222	745	104

When such difficulties have been located, remedial measures should be taken at once to replace incorrect responses with correct responses.

Another type of informal diagnostic testing that may be used in determining the combinations that are most in need of additional practice to bring them up to a desirable standard is illustrated by short selected lists of combinations like the following:

6+9=	7+8=	6+7=	9+3=	8+7=	6+5=
9+6=	9+7=	8+6=	7+6=	4+9=	8+5=
6+9=	8+8=	5+9=	7+7=	9+8=	7+9=
9+5=	6+8=	9+9=	8+9=	7+5=	5+7=

The pupils may be instructed to write the numbers from 1 to 20 down the left side of a sheet of paper, then when the tasks are read by the teacher at a certain rate the pupils may write down the answer to each task before the next combination is read. When the answer can only be guessed, the space should be left blank. Later, when the teacher reads the correct answers, all the blank spaces and wrong answers should be checked and the combinations and right answers written down for additional study, just as the words missed in an informal spelling test should furnish the material for further study. Frequent re-tests with similar lists enable the pupils to see how much progress they are making.

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  - Division 2 for grades 5 and 6
  - Division 3 for grades 7 and 8
- Scales R and S for the fundamental operations
  - Division 1 for grades 3 and 4
  - Division 2 for grades 5 and 6
  - Division 3 for grades 7 and 8

Scale C for addition
Division 2 for grades 5 and 6

Scale F for subtraction
Division 2 for grades 5 and 6

Scale K for multiplication
Division 2 for grades 5 and 6
Division 3 for grades 7 and 8

Scale N for division
Division 2 for grades 5 and 6
Division 3 for grades 7 and 8
Public School Publishing Company, Bloomington, Illinois.

Standardized Tests in Arithmetic

Clapp Standard School Test for Upper Arithmetic
Grades 5 through 8
University of Wisconsin, Madison, Wisconsin.

Courtis Standard Research Tests, Series B
Forms 1, 2, 3 and 4 for grades 4 through 8

Courtis Standard Supervisory Tests in Arithmetic
Test A for grades 4B, 4A, and 5B
Test B for grades 5A through 8A
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Monroe General Survey Tests in Arithmetic
Forms 1, 2, and 3
Test I for grades 3, 4, and 5
Test II for grades 6, 7, and 8

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Forms 1 and 2
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Test II for grades 6 and 7
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Test VI, carrying in multiplication
Test VII, zero quotient combination in short division
Test VIII, major difficulties in long division
Public School Publishing Company, Bloomington, Illinois.

CHAPTER III

THE ANALYSIS OF PUPIL DIFFICULTIES IN ARITHMETIC

In all groups of pupils there are those who are inferior in ability. The drill work given to classes as a whole seems to bring about but little improvement in those cases.

Courtis, Monroe, and others have shown the importance of adjusting the class procedure so that a pupil will be given work with the type of material that will best aid him in overcoming his weaknesses. The result has been the development of such standard practice materials as those prepared by Courtis and Studebaker. Yet, even with this material, some pupils fail to make much progress in their work. Special tests, by means of which the teacher can determine the processes or the precise part of a process in which the child is weak, have also been developed by Monroe, Osburn, and others.

There is still the problem of determining just what causes the child to fail in a certain type of work. What are the mental reactions of a child who is having difficulty? Are his methods of work correct? Does he use wasteful methods of work? What is the best method of discovering these weaknesses? How can they be overcome and eliminated?

In Minneapolis, a careful study has been made of pupils who are weak in the fundamental operations of arithmetic, as revealed by the results of standardized tests. It has been found that, in many cases, the difficulty is due to faulty habits of procedure in solving simple arithmetic

examples and problems. The faulty habits of individual pupils are ordinarily not detected by the teacher, when the entire class works as a unit, and they cannot be discovered from an examination of the test papers. Tests will always reveal certain pupils who are below the standard of their grade in ability. The problem before the teacher is to make the best use of these findings.

By means of careful study of individual pupils, and observation of how they work their examples and problems, teachers can discover the difficulties of pupils. This can be accomplished by having the work done aloud by the pupil so that the teacher can determine the procedure used by the child at each step. This requires tact and patience and a sympathetic attitude on the part of the teacher. With a little practice teachers can discover and correct many individual weaknesses. The *Courtis Practice Test* series is well adapted to this purpose. The manual that accompanies the tests contains descriptions of remedial work that can be done with pupils who fail on each of the tests. The series of carefully graded tests by Monroe and Osburn are also very helpful.

For the Minneapolis investigation by Miss Ella Probst, reported in the *Third Yearbook of the Elementary School Principals*, the summary sheet of difficulties in arithmetic described by Brueckner in the *Second Yearbook of the Principals' Association* was used. An improved method of discovering and recording these difficulties was also worked out. The exercises were mimeographed in carefully graded groups on a single sheet of paper.

The pupil began with one of the sets and proceeded as far as he could. This made it possible not only to preserve the complete record of a pupil's work, but the teacher could also see the precise place where the faulty habits first became evident. Notes showing the methods of work were written either in the margin or on the back of the sheet.

The difficulties that were detected were summarized for each pupil on an individual sheet, and the remedial work needed to overcome them was prescribed. The procedure in making such a study is as follows:

1. Select for study the pupils who fail in groups four or five on the *Courtis Supervisory Tests*, or who in other ways have shown weaknesses in the fundamentals.

2. While the other members of the class are at work at their seats, have one of the pupils in groups four or five go to the teacher's desk. This may be done before or after school. Tell the pupil that you wish to help him improve his work in arithmetic, and that you want to observe how he works his examples and problems, so that you can help him to overcome his difficulties. Encourage the pupil to be frank in his statements, and to give in full his procedure in working a problem with which he has difficulty. The teacher should be especially careful not to censure or criticize the pupil when peculiar methods of work become evident. It is important at this stage that these methods be discovered, and that remedial work be then undertaken to remove specific difficulties.

3. When the purpose has been made clear to the pupil, ask him to work a few simple examples and problems in each of the fundamentals and observe his general methods of work. It is a better plan to use some material especially organized for this work, similar to that described above. The teacher should especially note whether the pupil uses counting, tapping, excessive motor activity, or other faulty methods of work. Most pupils will quickly reveal their weaknesses. Many know what their individual troubles are. Some pupils have been found who are excellent in one fundamental process but poor in the others.

4. When difficulties are apparent, ask the pupil to work one of the examples or problems aloud. Encourage him to state in detail what his procedure is. Watch for faulty

statements, roundabout methods, and other types of difficulties that are described later in this chapter. Teachers will quickly discover that there are certain bad habits that are not apparent when the entire class works as a whole on a set of examples or problems, when it is not possible to study the work of individual pupils.

5. It is best to begin with the simplest processes, and then to proceed to the more difficult ones, in order to determine on just what level of work the pupil fails. It will be found in many cases that the most elementary combinations are not well established, and that drill work must begin there. Pupils of superior ability can be used to help in this work, but the teacher should check beforehand the methods which they use, and make certain that the pupils needing drill get the correct basic procedures.

6. It has been shown that arithmetic abilities are specific, that is, that each ability must be developed by special drill directed to that end. Arithmetic is a complex of many specific bonds and processes, and training in one procedure transfers to others only in so far as the same bonds are present. Superior ability in addition does not insure superior ability in subtraction, multiplication, or division. However, there has been found to be a fairly high correlation or relationship between the different abilities in arithmetic. Individual pupils are often found who, due to faulty training, have bad habits of work in only one phase of arithmetic work. These can be overcome when the deficiencies have been discovered.

To assist teachers to recognize these faulty habits, a partial list of some of the incorrect methods of work that have been discovered is given below. A brief description of the ways in which pupils do the work is given in each case. The remedial work that is necessary is determined by the responses of the individual pupil. Faulty methods must be replaced by correct ones. This can be done by

giving the child special attention for a short time so that the correct procedures can be established.

TYPES OF DIFFICULTIES IN ADDITION

9	1	6	8
1	4	7	9
5	6	9	2

1. Inspection to find starting point. Most of the pupils examined had no regular starting point. They would waste time inspecting the example to find out which combination they knew best. In the first example the response would be 9-10-15; in the second 6-10-11. When there were no easy combinations, the pupils would often begin with the hardest combinations in order to get these out of the way first. In the third example it is easier for the pupils to add 9 and 7, and then add 6 to this sum than to add in the reverse order. In the fourth example, many pupils first add 8 and 9 and then the 2.

6	4	1	4
5	6	3	2
3	2	8	6
			12

2. Starts backwards. In every case the examiner pointed to the left when she told the pupils where they were to begin; but when the word "Go" was given, several pupils began at the right. This is probably not a very serious difficulty, but it seems strange that boys and girls who have been in school five or six years should begin at the right end of a series of examples instead of at the left.

3. Skips around. Skipping around in the columns was quite a common failing. One wonders how pupils

who do this are able to get correct results, but some who use this method are quite proficient. One pupil proceeded as follows: $8+8=16$, and $4=20$, and $5=25$, and $3=28$.

Pupils who use the above method usually try to make the additions of the large numbers first.

3
8
5
8
4

4. Breaks up combinations. The breaking up of combinations was a common failing. Most pupils were able to give the sum of the first two numbers, even though they were large, as in the illustration; but some would either divide the 9 into a 7 and a 2, or into a 6 and a 3. If the former device was used, the response would be $7+7=14$ and $2=16$; if the latter response was used, it was usually found to be a case of a pupil who added by tens. This pupil would say $7+3=10$ and $6=16$. The pupil who adds by tens would say $9+7=16$, $16+4=20$, and $3=23$, and $3=26$, and $4=30$ and $8=38$.

4
8
3
7
9
7

This type of difficulty is very common. The pupils who use this procedure have not mastered the upper combinations.

3
0

5. Adds by tens. Many pupils look first for possible combinations of tens. In the illustration some pupils first picked out $4+5+1$ as a ten combination.

(4)
(5)
(1)
7
2

6. Carrying. It is hard for some pupils to remember what they have to carry, even in quite simple examples. Their attention is likely to wander. Usually, such pupils are able to succeed when they write down their answers as they work. Others are so intent on keeping in mind

45
36
72
153
(1)

what they have to carry that they are likely to give incorrect results. In the example in the margin one child obtained 153, but was so intent on remembering the 1 she had to carry that she gave the result as 151.

7. Trouble with the second addition. Some children know the combinations well, but they have trouble with the second addition. They know for example, that 8 and 4 are 12, and then they count when adding the 6. Courtis suggests that such a pupil be told to picture the 12 written on the paper directly under the 6 in position for adding.

6	
4	
8	

8. Bridging tens. Many children have difficulty in bridging the tens. They know the combinations, but are unable to succeed in column addition because they count all sums above 18. Pupils should be taught to see the numbers and to think the results, 42, 52, 62, only.

34	44	54
8	8	8

TYPES OF DIFFICULTIES IN SUBTRACTION

1. Subtracts the smaller. Some children take the number in the minuend from the number in the subtrahend, if the former number is smaller than the latter.

44
18
34

2. Zero difficulties. This type of example in subtraction is difficult for some children. In an example of this type one pupil thought of zero as nothing and borrowed from the 4 twice, once to make 13 and the second time to make 10.

403
36
277

3. Borrowing and adding ten to results. Many children seem to have difficulty with examples which involve borrowing. They often forget that they have borrowed and so their answers are 10 too large.

72
8
74

4. Roundabout methods of work in subtraction.

(*a*) Addition method.

$$\begin{array}{r} 7 \\ 3 \\ \hline 4 \end{array} \qquad \begin{array}{l} 3+3=6 \\ 6+1=7 \\ \text{Therefore 3 from } 7=3+1=4 \end{array}$$

(*b*) Evident lack of comprehension of the procedure.

$$\begin{array}{r} 37 \\ 15 \\ \hline 22 \end{array} \qquad \begin{array}{l} \text{15 goes into 30 twice.} \\ \text{Add the other 2 (obtained by taking 5 from 7).} \end{array}$$

(*c*) Combining addition and subtraction.

$$\begin{array}{r} 12 \\ 8 \\ \hline 4 \end{array} \qquad \begin{array}{l} \text{8 from } 10=2 \\ \\ 2+2=4 \end{array}$$

(*d*) One unnecessary step.

$$\begin{array}{r} 57 \\ 29 \\ \hline 28 \end{array} \qquad \begin{array}{l} \text{9 from } 17=8 \\ \\ \text{2 from } 5=3 \\ \text{1 from } 3=2 \end{array}$$

(*e*) Subtracting by counting 1 at a time.

$$\begin{array}{r} 21 \\ 4 \\ \hline 17 \end{array} \qquad \begin{array}{l} \text{1 subtracted leaves 20} \\ \\ \text{2 subtracted leaves 19} \\ \text{3 subtracted leaves 18} \\ \text{4 subtracted leaves 17} \end{array}$$

(*f*) Subtracting by counting 2 at a time.

$$\begin{array}{r} 22 \\ 8 \\ \hline 14 \end{array} \qquad \begin{array}{l} \text{1 two subtracted leaves 20} \\ \\ \text{2 twos subtracted leaves 18} \\ \text{3 twos subtracted leaves 16} \\ \text{4 twos subtracted leaves 14} \end{array}$$

Subtraction seems to be the most difficult of the fundamental operations for many pupils. Some pupils are not able to give the results promptly from inspection, and so they resort to all kinds of roundabout methods.

Some pupils have difficulty with particular combinations, which they can readily point out. One child failed wherever she had to use the 8 plus 5 combination. When she subtracted, she never gave the correct result, if the example involved this particular combination.

TYPES OF DIFFICULTIES IN MULTIPLICATION

1. The smaller number not always used as the multiplier.

43	$3\times2=6$		52	$3\times2=6$
2	$4\times2=8$		3	$5\times3=15$
86			156	

Some children do not use the smaller number as the multiplier. They may obtain the correct result but they seem to know certain combinations better than others. Almost invariably, such pupils use the larger number as the multiplier.

2. Zero difficulties. The first type of example caused the greater difficulty. Occasionally, a pupil places the zeros as in the second example.

103	511
3	30
339	000
	1533
	15330

3. Carrying difficulties. Here the pupil added the 6 from the 56 to 28 and obtained 34.

84	$7\times4=28$
7	
534	$7\times8=56$

Here the pupil added 6 to the 18. He also started his multiplication at the the left instead of at the right.

63	$6\times6=36$
6	
324	$6\times3=18$

The pupil was so intent on keeping in mind what she had to carry that she substituted the 8 for the 1 in writing the product.

$$\begin{array}{r} 29 \\ 9 \\ \hline 268 \end{array} \qquad \begin{array}{l} 9\times 9=81 \\ \\ 18+8=26 \end{array}$$

Some pupils are likely to reverse the numbers. In this case the pupil wrote 12 instead of 21 and failed to carry the 1.

$$\begin{array}{r} 31 \\ 7 \\ \hline 127 \end{array}$$

TYPES OF DIFFICULTIES IN DIVISION

1. Roundabout methods. One child said: 4 goes in 34, 8 times. $8\times 2=16$. Another reacted to this example in the following manner: "3 into 2, 0 times; 3 into 9, 3 times. No, that is not right, but 11 is too high, 10 is too high, 8 is too low, 9 is too low and 10 is too high. I don't know what to do."

$$\begin{array}{r} 16 \\ 2\overline{)34} \end{array} \qquad \begin{array}{r} \\ 3\overline{)29} \end{array}$$

$$\begin{array}{r} 9 \\ 3\overline{)29} \end{array} \qquad \begin{array}{r} 5 \\ 7\overline{)357} \end{array} \qquad \begin{array}{r} 9 \\ 5\overline{)545} \end{array}$$

In the first example above, the pupil paid no attention to the remainder. In the second the division was not completed. The pupil had become so intent on seeing how many times a certain number was contained in a two-place number that he failed to do even the first step in the third example correctly.

2. Zero difficulties. Some pupils habitually overlook the zero in working examples of this type.

$$\begin{array}{r} 21 \\ 3\overline{)603} \end{array}$$

3. Inability to subtract. Naturally the pupil who is unable to subtract readily or who uses roundabout or confusing methods of subtraction will have trouble with long division.

$$\begin{array}{r} 5 \\ 61\overline{)3477} \\ 305 \\ \hline 27 \end{array}$$

4. Forgets what the remainder is. Some pupils have difficulty in remembering what the remainder is. They are able to work the example, but when asked to do it orally, they fail.

$$\begin{array}{r} 49\frac{1}{3} \\ 3\overline{)138} \end{array}$$

This boy forgot that the remainder was 1. He thought it was 2, and worked the example on that basis.

5. Repeats tables for results. Some pupils seem to find it necessary to repeat the tables in order to obtain the results. For example, in $12\overline{)84}$ they think, 9, 18, 27. This same pupil stated that he was trying to think what table ended in 84. He said he thought $77+7=84$. On being asked whether he thought of any other tables he said he had thought of 99, but that was too large.

6. Difficulty with trial dividend. Several pupils failed in examples of this type because they had trouble in seeing the trial dividend. They failed on the example because they tried to tell the number of 3's in 39. $3\overline{)393}$

Teachers will note other difficulties when working with pupils. The prevalence of the types described has not been determined. It is not known which of them occurs most frequently in any one grade.

The list is not exhaustive. Teachers will find many other peculiarities in the arithmetic work of the pupils. If a principal finds peculiar types of responses, it is possible for him to check carefully the methods used by the teachers in order that the grade may be located where faulty habits begin. It has been found that when teachers become conscious of the strange methods of work of some of their pupils, they are more sympathetic and patient and make every effort to improve the conditions.

In one school, the principal took from their classrooms all pupils from grades 4B through 6B who were found to be deficient in their work in fundamentals. They were carefully studied by the principal, who took charge of teaching them individually during a short period of time. Rapid progress was made and soon some of the pupils were able to re-enter the regular class work. In the meantime the

regular teacher was relieved of the responsibility of teaching the backward pupils.

When one thinks of the peculiar difficulties that children have in spelling, reading, and in other subjects, the need of a hospital room for handicapped pupils is apparent.

AN INDIVIDUAL DIAGNOSIS

Difficulties of Clarence G. Clarence stated that he did not like arithmetic, and that addition and multiplication were especially difficult for him.

He finished the addition examples of the *Courtis Supervisory Test A* in one minute, and eight of the subtraction examples in another minute. During the allotted time of four and one-half minutes he finished the multiplication examples, working in all twenty-four examples, and made two errors in addition and one in multiplication. He was allowed to work until he had completed the division examples in the test. This took an additional three minutes and ten seconds. Only two of these examples were correct. One would be tempted to say, without further investigation, that most of his difficulty was due to division.

An examination of his work revealed several weaknesses. In the first example there were difficulties with subtraction. This was a great stumbling block to him, although this difficulty was almost overlooked by the teacher because the boy had a perfect score in his subtraction examples. In the second example he underlined the first dividend and then proceeded to neglect it entirely. The fact that the remainder was larger than the divisor did not trouble him at all. He obtained the correct answer for the fourth example, but he did not write down the last step.

Lip movement and vocalization were very evident throughout the processes. The examiner thought that Clarence also used his fingers in some way, but the boy denied this. Later, however, when asked how he obtained

an answer for a certain test exercise, he said that he counted by digging his finger nail into the edge of the card. He was willing to tell how he obtained all his results, and his willingness to co-operate aided greatly in discovering the many strange and faulty methods he used.

In addition, Clarence used roundabout methods. He did not add the addends in order. He needed drill especially on the upper combinations.

Upon closer examination of his work, in subtraction, it was found that although he had a perfect score in the subtraction problems of the test, he seemed to have difficulty with even the simplest examples. At first his difficulty appeared to be a matter of not knowing certain combinations, and after he had worked several examples, he was asked to tell how he obtained the following result. He had given the correct answer for this example but there was something strange in his method of procedure. He stated his method as follows: 15 goes into 30 twice. Add the other 2 (he obtained this 2 by taking 5 from 7). Two goes into 4 (3 and 1) twice and 1 over. He was asked to work other examples and he proceeded in the same way. He worked the examples 12 minus 8, as follows: 8 from 10 = 2. 2 plus 2 equals 4. He worked other similar examples in the same way.

$$\begin{array}{r} 37 \\ 15 \\ \hline 22 \end{array}$$

Clarence was asked to work a few additional examples in division. All went well as long as the examples were the reverse process of the multiplication tables, but when he came to an example like: 34 divided by 2, there was trouble immediately. He remembered his tables evidently, for he said 4 goes into 34, 8 times. $8 \times 2 = 16$ and he concluded that $34 \div 2 = 16$.

In working the example, 138 divided by $3 = 49\frac{1}{3}$, he did not seem to be able to retain the 1 in mind but he worked examples of this type quite readily with his pencil.

In working the following example in multiplication: he said:

32	$5\times3=15$
5	$2\times5=10$
115	

In working	63	he said: $6\times6=36$
	6	$6\times3=18$
	324	$18+6=24$

He said he always worked the examples this way, but when he was given his test paper he saw at once that he did not, and proceeded to work the next example. He stated immediately that he had made a mistake in the last example, that he had added the 6 to the 28 and so had obtained the answer 534.

$84\times7=534$	$7\times4=28$
	$7\times8=56$
	$28+6=34$

Difficulties in Reasoning Problems. A diagnostic study in which the Buckingham problems were used, was published by the Wisconsin State Department of Public Instruction. This study is based on 30,000 errors made by 6,000 pupils. It indicates that the most common difficulties encountered by the pupils were as follows:

(1)	Total failure to comprehend the problem	30%
(2)	Procedure partly correct but incomplete in one or two essential elements	20%
(3)	Ignorance of fundamental quantitative relations	10%
	Total due to inability to comprehend	60%
(4)	Errors in fundamentals	20%
(5)	Miscellaneous errors	2%

(6) Errors whose cause could not be discovered..... 18%

Total..... 100%

An analysis of the work of a 6A class of thirty-seven pupils in Minneapolis on the *Buckingham Arithmetic Test, Division II, Form I*, revealed interesting combinations of these difficulties. The mistakes the pupils made were divided into the classification shown in the table given below. The errors made on each problem are shown.

TABLE II

DISTRIBUTION OF DIFFICULTIES FOR EACH PROBLEM ON THE BUCKINGHAM ARITHMETIC TEST IN A MINNEAPOLIS GRADE 6A CLASS

Problem	Entirely Correct	Correct Method but Error in Fundamentals	Wrong Method but Correct in Fundamentals	Method Incomplete. Correct Fundamentals	Method Incomplete. Error in Fundamentals	Wrong Method. Error in Fundamentals	No. of Pupils Attempting each Problem
1	32	..	3	..	..	2	37
2	35	1	1	..	..	..	37
3	34	..	..	..	..	3	37
4	27	4	6	..	..	..	37
5	15	..	17	..	..	5	37
6	29	7	1	..	..	..	37
7	29	..	3	..	..	5	37
8	9	..	..	28	..	..	37
9	32	2	..	..	..	1	35
10	16	12	2	..	..	7	37
11	20	8	2	1	..	6	37
12	29	..	2	..	..	6	37
13	22	1	4	..	..	10	37
14	10	7	4	..	..	13	34
15	13	1	9	..	..	12	35
Total	362	43	54	29	0	70	548
Per Cent of Attempts	64	8	10	5	..	13	100
Per Cent of Errors	..	22	28	14	..	36	...

The table should be read as follows: Thirty-two pupils made no errors in solving problem 1. No pupils failed to solve the problem correctly because of the second type of difficulty, namely — Correct Method but Errors in Fundamentals. Three pupils failed on the problem because they used an incorrect method, though the work in the fundamentals was correct. No pupils failed on the problem for the next two reasons. Two missed the problem because both the method and the fundamental work were wrong. Thirty-seven pupils attempted to solve the problem. The results for the other problems may be read in the same way.

It can be seen that the reason for missing a problem varied with the problem. For example, problem 5 was missed by seventeen pupils because the method was wrong although the fundamental work was correct. Five missed it because of errors in both method and fundamentals. Problem 8 was missed by twenty-eight pupils because the method was incomplete although the fundamentals were correct.

At the foot of the table are given the totals for all problems for the work of the class. Forty-three problems were missed because of the first type of difficulty. The method used by the pupil was correct but there were errors in fundamentals. This was 8 per cent of the total number of attempts. Fifty-four problems were missed because of faulty method although the fundamentals were correct. This was 10 per cent of the total number of attempts. Of the total number of attempts, 50 per cent were wrong because the method was incomplete. In both method and fundamentals, 13 per cent were wrong.

It can be seen from these data that 8 per cent of the problems attempted were wrong because of errors in fundamentals alone and 28 per cent because of errors in method or in both method and fundamentals. This shows that in this group the big problem for the teacher is to teach the child correct methods of work. Of all problems that were

attempted, 21 per cent had mistakes in fundamentals, but 13 per cent were combined with errors in method, and may have been due to confusion on the part of pupils. The last line at the foot of the table gives the per cent of errors found in each group. The greatest proportion, 36 per cent, was found in the last type of error.

It is clear from these data that teachers must analyze the sources of errors made by their pupils and this must be done on the basis of what the individual child does with a problem. Where certain difficulties are found to be common throughout the group, teachers can give class work to overcome those difficulties.

It has been found that considerable work with problems without numbers has resulted in a much greater gain than by working only with problems containing numbers. Such problems can be handled as silent reading exercises and specific drills should be given to pupils to improve their abilities in the study type of reading.

Pupils should be taught to analyze problems according to the following steps:

1. What is to be found?
2. What facts are given which will aid in finding this?
3. How can I find the answer? What method shall I use in working the problem?

The Research Bureau of Ohio State University has prepared a set of diagnostic arithmetic reasoning problem tests by means of which teachers can locate the precise place of difficulty of a pupil in solving a thought problem.

Another comprehensive analysis of the causes of error in reasoning problems similar to those found in the *Buckingham Test of Arithmetic Problems* was published by G. O. Banting in the *Second Yearbook of Elementary School Principals*. His list of causes is as follows.

1. Failure to comprehend the problem in whole or in part. This may be due to any one of the following:
   a) Lack of general ability in silent reading.
   b) Lack of knowledge of the technical terms used, i. e., inability to read the language of arithmetic.
   c) Carelessness in reading. This often results in only a partial solution of the problem.
   d) Lack of the necessary experience to reproduce mentally the concrete situation of the problem.
2. Lack of the ability to perform accurately and readily the fundamental operations.
3. Lack of knowledge of facts essential to the solution of the problem.
4. Lack of ability to identify the proper process with the situations indicated in the problem. One may understand the processes very well and yet not know which to choose to solve a particular concrete problem.
5. Lack of sufficient interest in the problem to inspire the required mental effort.
6. Failure to form the habit of verifying the results.
7. The habit of focusing the attention upon the numbers and being guided by them instead of by the conditions of the problem. An example is the fact that many children feel that they must use in the solution every number stated in the problem.
8. Akin to the foregoing, pupils are sometimes completely nonplused by large numbers.
9. The habit of being guided by some verbal sign instead of making an analysis of the problem.

10. Lack of ability, or care, to properly arrange the written work in orderly, logical form.
11. The failure to recognize the mathematical similarity to type problems which the pupils understand, because of some unusual situation in the problem in question.
12. Lack of ability to understand quantitative relations such as cost, loss or gain, selling price, and the like.
13. The pupil may fail because the problem requires exertion beyond his span of attention.
14. The pupil may fail because of absolute inability to do reflective thinking.

Many of these difficulties can be overcome by specific remedial exercises which are designed to give practice in correct methods of procedure. Many of them are difficulties in reading. Many of them are, however, due to faulty basic habits and to an inadequate conception of the meaning and use of number. Many of the difficulties are due to lack of native capacity. The problem of the teacher is, in all cases, to provide the drill work and the class activity which will stimulate the pupils to well-motivated, worthwhile activity based on real life needs.

CHAPTER IV

SPECIAL STUDIES TO DETERMINE LIFE NEEDS

Some of the material of our present courses in arithmetic is out of date and is in the courses merely because it has been there for many years. The doctrine of formal discipline, which, as applied to arithmetic, assumes a transfer from the intensive study of arithmetic to other subjects, resulted in the inclusion and retention of certain difficult material which is beyond the present-day needs of the common activities of twentieth-century society.

The elimination of unnecessary material from the courses in arithmetic has received much consideration from educators during the past fifteen years. In 1904, McMurry suggested the need of such eliminations. Since then efforts have been made to discover what the arithmetic needs of people in life actually are. At first this was done by securing from various groups their opinions as to what should be eliminated from the curriculum. A study of this type was reported by Jessup and Coffman in 1914 in the *Fourteenth Yearbook of the National Society for the Study of Education*. In this report the content of the curriculum was discussed and the current practices were determined. The opinions of superintendents were secured, and on the basis of these opinions the recommendation was made that much of the useless material in the courses of study be eliminated.

Another study of this type, called *A Survey of Social and Business Uses of Arithmetic as Determined by Opinions of Business Men and the Arithmetic Problems Encountered by Parents in Daily Life*, was made by G. M. Wilson.

The conclusions of this investigation were as follows:

1. The business men of Connersville, Indiana, were in favor of eliminating the following topics from the courses in arithmetic: (*1*) troy weight; (*2*) apothecaries' weight; (*3*) longitude and time; (*4*) surveyor's table; (*5*) greatest common divisor; (*6*) least common multiple; (*7*) complex fractions; (*8*) compound fractions; (*9*) cube root; (*10*) foreign exchange; (*11*) compound proportion; (*12*) true discount; (*13*) compound interest; (*14*) partial payments; (*15*) partnership.

2. These same business men were in favor of more attention being given to the following topics: (*1*) the saving and lending of money; (*2*) building and loan associations; (*3*) mortgages; (*4*) modern banking methods; (*5*) the keeping of simple accounts; (*6*) investing money; (*7*) bonds as investments; (*8*) real estate as an investment; (*9*) the marks of a good investment; (*10*) public expenditures; (*11*) profits in different lines of business; (*12*) life insurance as a protection and as an investment.

3. Returns from superintendents of schools showed a strong tendency either to eliminate entirely or to give less attention to the topics mentioned by the business men as relatively impractical, and to give more attention to the modern economic and business application of arithmetic.

Wilson also collected and analyzed the arithmetic problems encountered by farmers, and showed that they usually have need for only the very simplest arithmetic. He suggested that, after the development of reasonable speed and accuracy in the fundamentals, and the mastery of the simple and more useful processes, the arithmetic work should be centered largely around those problems which furnish the basis for much business information.

Walter S. Monroe reported a study of the practical problems occurring in four different textbooks widely used in the schools. A practical problem was defined as a

problem which occurs in the course of a human activity. He divided human activities into classes: (*1*) occupational activities; (*2*) activities of the home; (*3*) personal activities; (*4*) activities of school children. He next compared the types of occupational problems found in the Federal Census Report. His conclusions were as follows:

1. Out of a total of 1,023 types of practical problems found in four textbooks, 720, or 71 per cent, occur in occupational activities.

2. No problems have been found in the textbooks which can be assigned to a large number of the specified occupations. These specified occupations make up 55 per cent of the total working population. If further research does not reveal a considerable number of occupational problems which come under these heads, a limitation of arithmetic as a vocational subject will be obvious.

3. The authors of these four textbooks are far from being in agreement on the type problems of arithmetic. Only one author out of four has recognized 511 out of 1,023 type problems, and 140 type problems have received the recognition of only two out of four authors.

4. If we take the judgment of the authors of these four textbooks as a basis for determining the importance of the problems in arithmetic, we arrive at the conclusion that there are not more than 372 type problems which are judged important enough to be included in a text by three out of four authors.

A study of specific arithmetic needs by Edwin Mitchell reported in 1917 in the *Seventeenth Yearbook of the National Society for the Study of Education*, is based on "Some Social Demands on the Course of Study in Arithmetic." He analyzed the arithmetic abilities required to use standard cookbooks, to handle the payrolls of a number of factories, and to understand sales advertisements and general catalogues.

The following conclusions are subject to the limitations of the data on which they are based. The data, do, however, furnish evidence as to some of the arithmetical activities of four groups of needs in the cases studied.

1. The great relative frequency of small numbers, especially of fractions and mixed numbers, is significant in its bearing on drill exercises in the form of problems.

2. It has proved worth while to ascertain the relative frequency of the different fractions and other kinds of numbers which are used in the arithmetical computations of a considerable number of people.

3. The prominence of the *dozen* as a unit of production and of trade suggests the importance of familiarity with the aliquot parts of 12.

4. Although decimalization is one of the distinguishing features of present-day arithmetic, *tenths* and *hundredths* play but an insignificant part as units of production and of trade in commodities.

5. Discount rates advertised in mark-down sales are most often expressed in per cents — the notable exceptions are the fractional forms $\frac{1}{3}$ and $\frac{1}{2}$ "off." Discounts are practically never expressed as so many cents on the dollar, in advertisements of mark-down sales.

6. The method employed in this study is feasible, and a part of the necessary procedure in the solution of the problem of what should constitute the course of study in arithmetic.

A study made of a somewhat different type of arithmetic needs is reported by Carl T. Wise, in the *Elementary School Journal* for October, 1919, "A Survey of Arithmetic Problems Arising in Various Occupations." A collection of 8,000 problems which parents were called upon to solve in their daily work was made by superintendents, principals, and teachers in high schools and in the sixth, seventh, and eighth grades. Problems were solicited from

schools in fourteen cities in the states of Wisconsin, Iowa, Illinois, Missouri, Texas, and California. His conclusions were as follows.

1. Of a total of approximately 8,000 problems classified from city and rural districts, representing problems that adults are called upon to solve, 85 per cent of the problems may be classified under the four fundamental operations and their combinations.

2. Differences in the classification of problems from city and rural districts are negligible.

3. A comparison of the classifications of problems collected from sections where interests are diversified, and from sections where interests are more specialized, shows that the types of problems are substantially the same.

4. This study shows that in actual daily practice people are seldom called upon to solve problems in compound interest, compound proportion, insurance, painting, plastering, masonry, bank discount, partial payments, foreign exchange, apothecaries' weight, alligation, compound or complex fractions, and troy weight.

5. The fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, and $\frac{1}{8}$ constitute 93.9 per cent of all fractions used.

6. Tables of measures and weights should be mastered for they are used continually in actual practice.

Clifford Woody reported an important study in the *Elementary School Journal* for March, 1922, on "Types of Arithmetic Needed in Certain Types of Salesmanship." The purpose of the study was to gain a reliable index of the types of arithmetic needed by the clerk in selling goods and by the consuming public in purchasing goods. This was done by analyzing bills of sale from hardware stores, department stores, grocery stores, and cash delivery statements. A similar study is reported by Charters. Woody's conclusions were as follows.

1. The amount of arithmetic actually used in the selling

of the goods as listed on bills of sale is very small. The problems commonly met on the four fundamental operations consist of:

a) Adding three-place or four-place numbers, having two, three or four addends.

b) Subtracting numbers with four places or less in the minuend.

c) Multiplying a three-place number by a two-place number.

d) Dividing a three-place number by a two-place or three-place number.

e) The fractions most commonly used are halves, fourths, sixths, and twelfths.

2. Multiplication and addition are used much more than division and subtraction. Subtraction is very little used except in making change, and then the additive method is commonly used. Problems show that there is a greater need for practice in our schools in multiplying by mixed numbers.

3. Greater emphasis should be placed on mental arithmetic as some stores utilize it almost exclusively. This means drill on the fundamental operations.

4. Denominate numbers are used in the buying and selling of goods.

5. Decimals and percentage are always used in connection with United States money.

6. Knowledge should be given concerning the use of measuring and computing devices.

Woody recommended the elimination of the nonessentials in arithmetic, and concentration upon the types of problems described above. Before all necessary eliminations can be wisely made, many other investigations involving other

phases of these same lines or other lines of business must be undertaken.

These studies make it clear that much is now being taught in our school work in arithmetic which is unnecessary because it is not used in life out of school. It should be noted that all these studies deal with adult needs. If this method is used as a basis for determining the content for the grades, difficulty will arise in determining what material should be assigned to a certain grade. In order that the course of study may be effectively reorganized, it will be necessary to study the activities that demand numbers in which pupils themselves engage out of school, and which would therefore be of vital interest in school. Many of these activities are undoubtedly similar to those of adults.

It is a fundamental principle of psychology that nothing should be taught unless there is a definite purpose in teaching it, and that this purpose is most effective when it is on the level of the child. Abstractions and mathematical generalizations of the types that are still included in many of our textbooks are on the adult level. Teachers should study the interests and needs of pupils in arithmetic, in order that additional motives may be discovered and used in teaching the subject.

Present Status of Selected Topics in Arithmetic. The courses of study in arithmetic throughout the country are being vitally affected by the numerous investigations that are being made in an attempt to vitalize the work in this subject, and to connect it more definitely with assured social needs. A national committee on the revision of the content of the course of study in arithmetic is also at work on the problem of determining, as scientifically as possible, what the content should be. The present status of twenty-four important topics found in most courses of study, and a comparison with their status in 1909, at the time of Wilson's study, and in 1915, at the time of Van Houten's

study, are given in the following table taken from the report by Guiler, published in the *Educational Research Bulletin*, Ohio State University, February 6, 1924, Vol. III, No. 3.

TABLE III

GRADE PREDOMINANCE OF ARITHMETICAL TOPICS

Arithmetical Topics	Grade Predominance			
	Van Houten's Study	Wilson's Study	Payne's Study	Guiler's Study
Notation	3	1, 2, 3	1, 2, 3	2, 3
Numeration	3	1, 2, 3	1, 2, 3	2
Forty-five Combinations	2	.....	.....	2
Multiplication Tables	3	.....	2, 3, 4	3, 4
Addition	2, 3, 4	2, 3	1, 2, 3, 4	2, 3
Subtraction	2, 3, 4	2, 3	2, 3, 4	3
Multiplication	3, 4	2, 3, 4	2, 3, 4	4
Long Division	3, 4	4	4	4
Fractions	5	4, 5	4, 5, 6	5
Decimals	5, 6	5	6	6
Denominate Numbers	6	4, 5, 6, 7	4, 6	5, 6
Percentage	6, 7	6, 7	7	6
Interest	7	7	7	7
Commercial Discount	7	.....	.....	7
Profit and Loss	7	7	7	7
Commission	7	6, 7	7	7
Tariff and Duties	7	7	7	7
Stocks and Bonds	8	8	7	8
Insurance	7	6, 7	7	7
Taxation	7	7	7	7
Money and Banking	8	7	7	7, 8
Mensuration	6	6, 7	8	8
Involution and Evolution	8	8	8	7
Ratio and Proportion	8	6, 8	8	8

The striking feature of this table is the remarkable agreement of the four studies in the grade placement of most of these topics. Guiler reported also the modal, or most frequent, grade occurrence of various abilities involving integers as he found them in thirty up-to-date courses of study. These data are given in the following table.

TABLE IV

THE ADJUSTMENT OF ARITHMETICAL OBJECTIVES TO PUPIL MATURITY WITH SPECIAL REFERENCE TO INTEGERS

Arithmetical Objectives Related to Integers	Modal Grade Occurrence
Reading Arabic Numerals: Ability to	
1 Read to 100	1
2 Read to 1000	2
3 Read four to six-place numbers	4
Reading Roman Numerals: Ability to	
1 Read through 12	2
2 Read through 50	3
3 Read through 100	4
Writing Arabic Numerals: Ability to	
1 Write to 100	1
2 Write to 1000	2
3 Write four to six-place numbers	4
Writing Roman Numerals: Ability to	
1 Write through 12	12
2 Write through 50	3
3 Write through 100	4
Addition: Ability to Add	
1 The first 45 combinations	2
2 Single columns, 4 to 6 addends	2
3 Three to five-place numbers, 3 to 5 addends	3
Subtraction: Ability to Subtract	
1 The first 45 combinations	2
2 Three to four-place numbers	3
3 Five to ten-place numbers	4
Multiplication: Ability to Multiply	
1 Repeat the multiplication tables	3
2 Multiply: Multiplicand through 4 orders Multiplier through 3 orders	4
Division: Ability to Divide	
1 Repeat the division tables	3
2 Divide: Dividend 3 to 6 places, divisor 2 to 3 places	4

It should be clear from the investigations here described that a good beginning has been made in the development of scientific methods of determining the content of the courses in arithmetic. Most of the studies are concerned with present conditions and can not be considered to represent the ideal. Much must be done to determine

more adequately just what the social needs of arithmetic are and what they should be. This must be done by studying both child and adult activities, and selecting from these activities a content which will satisfy the demands of everyday life and at the same time give the child those interests and meanings that can later be developed and expanded from the point of view of their use in the vocations and professions.

Time Allotments to Arithmetic. The amount of time given to arithmetic in different schools varies widely. The table below shows the median time for cities of different sizes and for rural schools as reported by county superintendents.

TABLE V

MEDIAN MINUTES PER WEEK GIVEN TO ARITHMETIC BY CITIES OF VARIOUS SIZES AND BY RURAL SCHOOLS AS REPORTED BY COUNTY SUPERINTENDENTS

Grade	4,000 to 10,000	10,001 to 20,000	20,001 to 50,000	50,001 to 200,000	200,001 and over	Reported by Co. Superintendent	630 Schools
I	80	52	80	80	50	51	75
II	95	89	102	112	100	63	100
III	125	122	150	150	130	75	125
IV	135	126	150	175	144	...	150
V	152	148	150	180	155	100	150
VI	155	148	150	200	170	100	150
VII	155	148	150	186	175	107	150
VIII	175	173	150	150	200	125	164

Much less time is reported by county superintendents than is allotted in the various cities and towns. There is, in general, an increase in the amount of time reported from the lower to the upper grades, except in cities from 50,000 to 200,000, where the maximum of 200 minutes is reached in grade six. Teachers who use the median amounts for

the 630 schools listed in the table, can be sure that they are following the general tendency of present practice. Teachers should always feel free to adjust the class time to the needs of the group.

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PART TWO

CHAPTER V

DRILL IN ARITHMETIC

With the development of the theory of formal discipline, arithmetic, like other practical subjects of instruction, took on a new value — that of training certain faculties, especially the so-called faculties of memory and reasoning. So long as drill was considered of great value as a means of training the memory, the materials used and the methods employed mattered but little. The absurdity of much of the material commonly used in arithmetic received no special attention. However, with the introduction of a larger number of subjects into the elementary school curriculum, the importance of arithmetic came to be challenged. Following the breakdown of the theory of formal discipline the content of the courses in arithmetic began to be actively assailed. The emphasis upon the interests of the child began to determine the methods of teaching the subject. Several investigations revealed the wastefulness of time spent in teaching many of the arithmetical processes.

The use of such measuring instruments as the *Courtis Standard Research Tests* has done much to keep the standard of efficiency in the four fundamental operations at a fairly high level, notwithstanding the protests against the deadening effects of uninteresting drill.

As a result of the many scientific studies that have been made in the teaching and learning of arithmetic, the content

of the subject has now been made to conform fairly well with the social and business needs. Recent analyses of the content from a psychological point of view have shown, in part at least, the relative difficulty of the various combinations and processes in arithmetic. Intensive studies of the textbooks and practice exercises have shown clearly where new emphasis needs to be placed in order to properly adjust the amount of practice, and to shift it to those particular parts of arithmetic where it is most needed. Furthermore, psychological analyses of the mental functions involved in the learning of the arithmetical operations are beginning to have their effects upon the methods of teaching arithmetic.

Arithmetic, a Habit-forming Study. Although the problems in arithmetic are now being stated in such a way as to make the study contribute to the pupils' ranges of information regarding the business and social activities of their environment, arithmetic is not considered primarily an informational or content subject. It involves the formation of a large number of very specific habits as well as more complex habits or co-ordinated methods of attacking tasks or problems developing out of the inter-relations of the simpler habits. For this reason arithmetic is fundamentally a habit-forming process and depends for its economy in development and efficiency in functioning upon conforming to the laws of habit formation.

Formal Versus Rational Drill. To many people drill carries with it the implication of repetition alone, without any specific reference to the use of the materials learned or any felt need for the drill. Less than half a century ago the first step in learning to read was the learning of the letters of the alphabet in sequence by repetition. Music was begun by teaching the pupils to sing the scale; the children were told that if they learned the scale well, some day they would be permitted to sing songs. In draw-

ing classes, the children first learned to make horizontal, vertical, and oblique lines which were later combined into geometric figures. The combinations in arithmetic were learned in tables until they could be recited glibly. Thus the method of instruction accepted and used only a few decades ago involved the practice and mastery of the habit elements of each subject without reference to their uses, and as embodied in a type of organization that had little or no reference to the situations in which they might be used.

This kind of drill was carried on with the illusive hope that the mastery of these elements in the most formal organization or even in isolation would make their use effective in case a need for them might arise. The principal motive back of such work was the requirement set up by the teacher; the manner of instruction was intrinsically uninteresting as few variations were employed.

The chief characteristic of this type of instruction, which is known as formal drill, is that the connections or facts are exercised in isolation or else in a formal organization that has little or no relation to their practical use. This sort of instruction fails to meet even the law of exercise, for that implies that the habits will be practiced in the ways in which they will be used and that elements will be put together that belong together in real situations. No provision whatever was made for arousing the feelings of achievement or satisfaction demanded by the law of effect.

Rational Drill. In marked contrast with this type of instruction is the emphasis which is now being placed upon the project method in teaching, the socialized recitation, and purposeful activity. Teachers are urged to teach by projects that lie within the range of the pupils' interests. Today practically every adequately trained teacher of reading begins with short sentences and phrases that tell something of interest to the children instead of starting

with the alphabet; the modern teacher of music begins her instruction with rote songs; the skillful teacher of drawing encourages the children to make free-hand pictures at the start, crude and imperfect though they may be; and the progressive teacher of arithmetic uses problems stated in clear and simple language that are related to the experiences and interests of the children, in place of the artificial and intricately worded exercises of the earlier texts.

The kind of practice in which the pupil has some purpose in view for the facts he is learning and in which, in the learning process, the inter-relations of these facts are emphasized is very different from the formal drill exercises of earlier years. Today *6 times 7 are 42* may be learned because a need is felt to know how much six 7's make in counting quickly where the objects are orderly arranged or thought of, and also where the pupil learns that 42 is 7 more than 35. This tendency has brought the process of instruction more into harmony with the nature of the learning process. It has done much to replace routine, monotony, and mere disciplinary training with interest, enthusiasm, and purposeful activity.

The emphasis today is being placed upon the things that are interesting to children but this does not mean that habit formation, practice, and drill have been abandoned or neglected. On the contrary, they too are used purposefully. In reading, for example, the pupil gets the necessary drill on words by reading much and widely; in music, the necessary drill is obtained by learning many songs; in drawing, by drawing many different things; in arithmetic, by solving many but varied interesting problems involving the same processes and operations. Again, just as in music attention is frequently called to some bit of technique where improvement can be effected and songs illustrating this and giving opportunity for practice appear from day to day, so in arithmetic the pupils are assisted through the

use of tests to locate their particular difficulties and then practice materials are provided for overcoming such obstacles as stand in the way of the ready solution of the problems of everyday interest and value. Furthermore, in this practice material, which is introduced where a need for it has become apparent, the laws of habit formation are skillfully embodied to produce the maximum results.

The Law of Exercise. One of the earlier but imperfectly recognized laws conditioning the formation of habits is what Thorndike has called the "law of exercise." This law asserts that the more often an act is repeated under conditions that are not annoying, the stronger the habit or stimulus-response connection becomes. By "stronger," is meant that the response comes to follow the stimulus with more certainty and with greater quickness. The law does not imply improvement in the sense of eliminating waste motions or unnecessary steps, nor does it imply improvement in the sense of building up inter-related and integrated habits. Furthermore, it does not imply improvement of any sort when the repetitions take place under annoying conditions.

With a sufficient number of repetitions useless ways of doing things or intermediate steps in a habit formation process may become firmly fixed. By mere repetition without annoyance a habit, like saying 9 and 7 are 16, 16 and 8 are 24, 24 and 7 are 31, may become stronger and stronger instead of developing into the more economical habit of thinking 16, 24, 31, when adding a column of numbers, like, 9, 7, 8 and 7. However necessary repetition may be, practice makes perfect only in the sense that it strengthens whatever connections are being formed without annoyance or with a feeling of satisfaction, regardless of whether the habits are the most useful ones, simply intermediate steps, or even interfering or positively detrimental. A pupil may learn to think at once, 63, when he sees a

situation calling for nine 7's, or he may learn to first repeat the 9 and 7 aloud or he may learn to think or say 56 when he sees a 9 times 7 situation, if he started out with a wrong response.

The Law of Effect. Just as important as the law of exercise, is the process which Thorndike has called the "law of effect." This means that in some way the condition of the nervous system accompanying a feeling of satisfaction has a definite influence upon the permanency of the brain cell connection formed under its influence. The feeling of satisfaction may come as a result of seeing that some gain has been made, as in being able to recall more readily some difficult combinations that have previously been somewhat elusive. Again it may come as a result of the successful functioning of the preparatory processes in the working out of an answer to some problem. In the first case a greater degree of confidence is aroused by the successful recall, which seems to be needed for a feeling of surety as well as for actual surety. In the second case a greater degree of organization seems to have taken place in the connections, which makes the successful attack on similar problems easier.

The law of effect is at work when the pupil has a purpose in view, whether that purpose be to outstrip his neighbor, to surpass his own previous record, to work out the solution of a problem that challenges him, to attain a goal that he has set up, to satisfy his curiosity as to how a process really works, or the feeling of the worth of the process to be learned.

To stimulate the pupil to purposeful activity in acquiring the arithmetical combination habits and operation habits is quite as essential as to stimulate him to acquire information in the content subjects. That this can be done, and that the results of practice carried on under stimulating conditions are unusually effective, is indicated by the

findings of Kirby previously referred to, where in some fifty to sixty minutes of motivated practice under experimental conditions, a relatively permanent gain of from 48 per cent to 75 per cent of the original ability was attained in the rate of adding and doing short division operations. Similar results are evident in many experiments where there was an urge to achieve results in the discovery of short-cuts or a better organization of habits.

Effective Drill Must Be Planned. Effective drill in arithmetic, that may be expected to bring this feeling of satisfaction, must be intelligently directed practice. It must be done with enthusiasm by the pupils, and it must not be continued until the zest is lost. By experiment, J. C. Brown found that a five-minute period of practice on the simpler combinations at the beginning of the arithmetic period, for an interval of only twenty days, resulted in marked improvement not only in the mastery of the fundamental operations in arithmetic but also in the solving of problems. Such drill cannot be indulged in, however, at the beginning of each arithmetic period just as a matter of course, any more than it can be kept up long, if the best results are to be obtained from it.

Definite plans for arousing a feeling of satisfaction with what has been accomplished as well as adequate stimulation must be provided. In the case of the simpler combinations, for example, each pupil may keep a record of the number of combinations on a series of mimeographed practice sheets done correctly each day in a two, three, or four-minute period of practice at the beginning of the class period. The attempt to surpass his own record of the preceding day or preceding week furnishes a very real and wholesome stimulus, and when the aim is attained, it affords a feeling of achievement and joy.

Effective drill on the more highly organized processes, such as the column addition of different place numbers,

or the multiplication of a three-place number containing a zero by another three-place number containing a zero, involves the isolation by testing of any specific processes that may be giving difficulty, such as neglecting empty spaces, adding a seen number to a thought-of number, placing the numbers in the proper place in the result, and the like.

Better Distribution of Practice Needed. It is very important that the more difficult combinations get a sufficient amount of practice. As already pointed out in some of the scientific studies, outlined in the first chapter, the easier combinations like $2+1=3$, $3-2=1$, $2\times2=4$, and $4\div2=2$, receive by far the greater amount of practice in the tasks contained in most textbooks in arithmetic, while such difficult tasks as $7+9=16$, $9-7=2$, $6\times9=54$, and $72\div8=9$, receive a relatively small amount of practice.

The selection of the particular combinations on which practice is most needed is quite as important as the isolation of difficulties with the various operations. It is at this point that the diagnostic tests become essential. Combinations with which pupils have special difficulty may be studied in much the same way that words missed in an informal spelling test are used as a study list.

In the practice on these difficult combinations care must be taken that the lists given are not too long, that the drill is distributed over several days or weeks, and that some provision be made for drill on the combinations whose answers are not thought of quickly and accurately. This does not mean that any combination should be practiced in isolation. All related associations should be used. The pupil in learning that $6\times9=54$ should also learn such combinations as $5\times9=45$, $7\times9=63$, and so on.

However important quickness in thinking of the answers may be, accuracy is still more important. The degree of inaccuracy that Kirby found in the case of fourth-grade

pupils — about 80 per cent — could have little actual value in everyday life. The accuracy standard set by Thorndike of 996 or 997 out of each 1,000 responses may seem high in comparison with actual achievement in school, but not in comparison with commercial requirements. In fact, it is only when the combinations become relatively well fixed, and the responses possess a high degree of accuracy, that the pupil is ready to take up successfully the more difficult operations.

When a new operation is taken up for the first time it is wisest to have it formed as a habit of response rather than through a detailed explanation which pupils are expected to repeat. In the addition of two-place numbers involving carrying, such as those shown to the right, the pupil should form the habit of putting down the 2 and adding the 1 to the digits in the second column in many different tasks rather than being required to explain the process as ten units of one order make one unit of the next higher order. When a new operation is thus to be learned the early tasks should involve the simplest and most familiar combinations, so that the attention of the pupil will not be diverted from the new element to some combination difficulty. After a little practice, the more difficult combinations should be used more often in the operations than the simpler ones.

$$\begin{array}{r} 14 \\ 13 \\ 15 \\ \hline \end{array}$$

Provision for Proper Organization of Habits. The organization of specific habits involves something besides the specific habits themselves. The sequence in which the habits occur, or the pupil's method of attack, is an important element. Whether in the addition of several one-place numbers, like those shown to the right, the pupil starts at the top and thinks 6 when he sees 2, 4, then thinks 13, 19, and finally 22, or whether he first looks for the combinations that make 10 — the 4 and the 6, the 7 and the 3 — then multiplies

$$\begin{array}{r} 2 \\ 4 \\ 7 \\ 6 \\ 3 \\ \hline \end{array}$$

the 10 by 2 since he has two 10's, and adds in the additional digit, making 22, makes a considerable difference in his probable chances for an accurate answer as well as in the amount of time which he consumes in getting the answer. As the tasks become more complicated the opportunities for variation from a reasonably effective sequence of simple habits becomes greater and greater.

On the other hand, a more complete organization of simple habits may involve the slurring over or omission of many of them in the final organized response to such a task in multiplication as that shown here in which the first product may be thought of as 1,284 without a conscious going through of any of the intermediate steps of thinking $4\times1=4$, $4\times2=8$, and $4\times3=12$. In the same way, the second product of 963 may be thought of as a unit and practically automatically written under the 1,284, as in the example above. The sum of these two products may then be seen to be 10,914 without any definite awareness of carrying. While this latter procedure may represent an organization of habits not met with in the elementary school except in the case of the gifted pupils, it does stand at the opposite extreme from the procedure illustrated in the hunting for the 10's, finding their sum, and then adding the 2. Between these two extremes are to be found all stages of consolidation of habits and operations from cumbersome sequences to effective organizations.

$$\begin{array}{r} 321 \\ 34 \\ \hline 1284 \\ 963 \\ \hline \end{array}$$

Variety in Drill Necessary. Important as is proper repetition in forming the combination habits and in forming the more highly organized responses, the need for variety in drill is equally important for two reasons. Without variety the practice work is likely to lose its appeal and thus become monotonous and unaccompanied by the feelings of achievement and satisfaction that seem so essential to economy in learning.

In order that no operation may become firmly associated with some unessential or misleading element, any particular operation should be made in connection with changes in all the elements except the fundamental ones. A few illustrations will clarify this need. In case pupils are first taught to carry in situations in which the number to be carried is 1, there is danger that the carrying operation will come to be thought of as merely adding 1 to the sum of the numbers in the next column. In teaching the simple number combinations it is not sufficient to teach that 2 marbles and 3 marbles make 5 marbles, but it is also necessary to divorce the tendency of thinking 2 and 3 make 5 from marbles by teaching that 2 blocks and 3 blocks make 5 blocks, that 2 dogs and 3 dogs make 5 dogs, and the like. In developing the 2-and-3-are-5 habit the connection needs to be established in a variety of symbolic situations also, as $2+3=5$, $2+3=5$, 3 and 2 are 5, so that the response does not become connected with some unessential feature like the use of the plus sign (+) or the equality sign (=).

$$\begin{array}{cc} 2 & 3 \\ \underline{3} & \underline{2} \\ 5 & 5 \end{array}$$

A perception of the nature of a process or operation, like adding or subtracting, comes not from using the same combination over and over but from using the same operation with many different combinations. As pointed out by Thorndike, the long continued use of whole numbers in multiplying and dividing leads to the formation of the association that multiplying always gives a bigger quantity and dividing always gives a smaller quantity — concepts that do not hold true when multiplying or dividing by proper fractions.

The use of the same processes in a variety of situations is just as essential when dealing with problems as in dealing with the operations themselves. The use of *dozens* and fractional parts of *dozens*, for instance, should not be confined to problems involving the buying and selling of eggs

alone. It is important that problems involving the buying and selling of oranges, bananas, handkerchiefs, buttons, collars, and the like be introduced and that the use of them be not even restricted to problems involving buying and selling but extended to any problems that might arise in normal business or social situations, as in having four dozen drawing pencils distributed to a class.

Verifying Results as Drill. One of the most effective forms of drill in arithmetic is that furnished by verifying of results. This provides a stimulating motive. At the same time, it creates a natural situation calling for accuracy, in which satisfaction is afforded by correct answers and a natural source of annoyance penalizes the individual for wrong answers. It also furnishes a stimulus to the pupil to try to locate his own difficulties. The use of diagnostic tests in this connection offers an opportunity for achieving improvement that will be seized upon by the earnest pupil. Not only will the pupils be concerned with locating their difficulties but also with detecting the conditions responsible for them. Once this stage of motivation has been attained the function of intelligently guided drill becomes apparent to the pupils. It is no longer an arbitrary requirement of the teacher but a rational method of overcoming obstacles that hinder progress.

Use of Practice Exercises. Another method of securing more effective drill is through the use of practice exercises, several of which are now available. The *Courtis Practice Exercises*[1] and the *Studebaker Practice Exercises*,[2] which have been in use for some years, have furnished teachers of arithmetic with inexpensive yet effective devices for securing drill on the fundamental operations. The practice exercises more recently devised by W. J. Osburn[3] not only

[1] *Courtis Practice Exercises*, World Book Company, Yonkers, N. Y.
[2] *Studebaker Practice Exercises*, Scott, Foresman and Company, Chicago.
[3] *Osburn Practice Exercises*, Houghton Mifflin Company, Boston.

provide excellent teaching situations, but also furnish a greater degree of equality in the distribution of practice among the various combinations than is likely to be attained in any other way. Such practice exercises may be employed to set a reasonable goal of attainment for each pupil. They likewise provide a means of motivating the drill work in a rational way.

Place of Experiments in Drill. Every teacher of arithmetic has her favorite methods of providing for drill, yet few teachers subject their methods to any experimental tests to determine the relative effectiveness of such methods as compared with other methods. Such a procedure may be expected to enliven the work not only of the teacher but of the pupils as well, as was demonstrated in the experiment by Kirby. The slight amount of extra work that is involved in such a simple experiment is more than compensated for by the added interest in the work. Both pupils and teacher are provided with a fresh incentive that cannot fail to be stimulating.

For an effective classroom experiment, the procedure should be kept simple, the time relatively short, the tests at the beginning and end of the experiment easy to administer and quick to score. The experimental practice should be attractive to the pupils and appeal to their interest. The correcting of the experimental exercises should be of such a nature that it may profitably be done by the pupils. A purpose is furnished for the pupils to learn how to plot their own individual scores and gains made during the experiment as well as the gains made by the class when working under the experimental conditions. An example of such an experiment is the one carried out by J. C. Brown in which several classes took part. The co-operation of several teachers in the same experiment gave added interest to it.

Immediately after the first test was given, half of the

classes examined were given drill upon the fundamental operations for five minutes each day. The first five minutes of the recitation period were given over to this drill, which was partly oral and partly written, with the time about evenly distributed among the four operations of addition, subtraction, multiplication, and division. The teachers in the drill sections emphasized both speed and accuracy while the teachers in the non-drill classes had no special drill upon the fundamental operations during the period of the experiment. In order to keep all other conditions except the drill identical the same daily assignments in the textbooks were covered by all the classes. At the end of thirty days a second test was given. This indicated that the drill classes made from two and one-fifth to three and one-tenth times as much improvement in the amount of work done as the non-drill classes, and at the same time gained 11.7 per cent in accuracy, while the non-drill class made no gain in accuracy. Even after a vacation of twelve weeks a third test revealed the fact that the superiority of the drill classes was maintained over the vacation period. Such experiments not only add to the teachers' knowledge of their work, but give an added purpose to the efforts of the pupils.

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See also *Suggested Readings* on page 183.

CHAPTER VI

STANDARDIZED DRILL MATERIALS

To assist teachers in providing well-organized drill on the fundamentals for the pupils in their classes, standardized drill materials for use in grades four to eight have been prepared and are now widely used. The best known are the drills constructed by Courtis and by Studebaker. These exercises each consists of a series of drills with whole numbers placed on separate cards. Each card contains materials of one type, such as three-place addition or simple division. The pupil begins with the first card and when he has been able to complete the work on that card within the time allowed for his grade and with perfect accuracy, he can proceed to the next and so on until he has completed the entire set of cards. The work is so arranged that the material on each succeeding card is increasingly difficult, and at the same time gives drill on the materials previously studied. One new step is added at a time. A pupil can progress as rapidly as he is able to complete the work of each lesson. Some pupils finish the entire set of exercises during the first month, while others are so superior in their achievements that they are excused from all the drill work.

The principles underlying these practice lessons are as follows:

1. Children do not all need the same amount of drill on the different phases of the fundamentals. Some children, for example, are superior in addition and do not need drill, while others may be weak in this fundamental and must do extra work to reach the desired standards.

2. Children vary in the rate of improvement resulting

from drill. Some children will very quickly reach the desired standard in one phase, while others will require many more periods of drill.

3. When a child has reached the desired standard in one ability, he should be allowed to undertake work to develop his ability in the next higher level to the required standard. Under the usual procedure in mass instruction, the pupil up to standard in one ability is frequently held back and is required to do unnecessary drill with the rest of the class. This wasteful procedure is eliminated in classes in which work is done with materials properly organized.

4. The child should know the status of his abilities and should have some means of determining the goals to be reached. This gives purpose to his work and serves as a powerful stimulus to effort to bring about improvement. Each lesson in the Courtis or Studebaker materials represents a definite goal, and the child knows when he has reached it.

5. The teacher should be free to work with those children who are encountering difficulties. This cannot be done in the ordinary mass instruction.

6. Drill work to be successful must be based upon previously well-established habits. If a child is required to pass on to work of a higher level of difficulty before the abilities needed to successfully undertake it have been developed, there can be but little progress. The Courtis materials are organized in such a way that this difficulty is eliminated.

7. The rate at which a child can solve his examples is a measure of his skill, but accuracy must not be sacrificed to gain speed. Life demands that an individual be fairly rapid and very accurate in computations. These drill materials emphasize both speed and accuracy.

8. The rate at which a child can compute should increase grade by grade as the child reaches higher levels of

maturity. For this reason the amount of time given to the pupils to accomplish the work of each practice lesson is reduced with each succeeding grade. The amount of time for each card of the Courtis tests ranges from $6\frac{1}{4}$ minutes in grade 4B to only 3 minutes in grade 8A. The time for all cards is the same, except for some of the last lessons, where the time is doubled.

9. Children who are up to the standard for their grade should be excused from all drill work.

10. The amount of time given to drill each day should be reduced to a minimum, and there should be short periods of intensive work at frequent intervals. No teacher need take more than ten to fifteen minutes a day to give a class a lesson using the Courtis or Studebaker materials. The time in class should, as far as possible, be devoted to accomplishing the chief purpose of arithmetic work, namely, training in quantitative thinking.

11. Drill work should be individualized, but the method of doing this must be adapted to our present system of large classes. With these standardized drill materials teachers can handle classes of forty or fifty pupils.

Detailed directions for using the drill materials are given in the teacher's manual issued by the publishers. The manual also contains suggestions as to the best methods of procedure that can be used to help pupils who are encountering difficulties. The results that can be secured with standardized drill materials have been found to be much superior to those where the work is done under the usual plan of mass instruction, where but little attention can be given to the needs of the individual pupil.

For further information regarding the topics treated in this chapter see the *Suggested Readings* for Chapter V, page 82.

CHAPTER VII

WASTE IN ARITHMETIC

This is an age of great commercial and industrial activity. The invention and improvement of numerous labor-saving devices, the improved facilities for transportation and communication, and the significant discoveries in numerous fields of scientific research have revolutionized commercial, industrial, and economic conditions within the last few decades. Efficiency has become the watchword, and efforts are being made to eliminate all possible waste of time, energy, and materials. The efficiency engineer is a product of modern conditions.

The progressive educator of today is interested in any investigations which seek to establish standards of efficiency that may be applied to the schools. Within recent years, several school systems have been surveyed and their efficiency has been judged by experts who had no immediate or personal interest in them. In so far as the equipment, organization, and instruction of the schools can be standardized and the efficiency measured, such investigations, when conducted with a genuine desire to improve existing conditions, are of great value.

It is a comparatively easy matter to establish standards of excellence for many material objects, and to classify more or less accurately the quality of a given product when compared with a standard product of the same kind. It is vastly more difficult to set up standards of efficiency for mental products. Some phases of school organization, equipment, and management can be measured with considerable accuracy when compared with recognized stand-

ards. Some of the products of the school cannot as yet be measured as accurately as we wish, but this should not deter us from applying standards of greater or less refinement to those that can be measured.

The progressive educator looks with favor upon all attempts to refine the standards of measurement now in use, and will welcome the establishment of other appropriate standards for the measurement of school activities and products. The teacher should constantly seek to establish a better balance between effort and result, a closer adjustment of means to ends. An investigation of subject matter or of method in a given field that results in the elimination of waste and in the discovery of ways and means of securing a better balance between effort and result is justified.

This chapter is devoted to a consideration of some of the ways in which waste can be eliminated in the teaching of arithmetic. The topic will be considered from the point of view of subject matter and of methods of instruction.

Before a manufacturer can determine the percentage of waste involved in making a given product, he must have in mind the exact product that is to be made. If he wishes to determine the percentage of waste material, he must investigate the extent to which the output balances the raw material. There must be a consideration of means and of ends. A better adjustment of means to ends always means an economic gain.

If we wish to investigate the sources of waste in arithmetic we must first establish as definitely as possible the aims in teaching the subject. Unless the goals which are striven for are known, the degree to which current practice enables us to reach the goals cannot be determined. The degree of closeness of approach to the ideals to be reached is a factor of great importance.

We would justly criticize the manufacturer who had no

clear idea of what he was attempting to produce. In the industrial world an efficient manager knows what he is attempting to do, and he sees the necessity for each step in the process. Each workman strives to accomplish a definite end. The part of the work that he does may be but a small part of that necessary to make the finished product, but it is the duty of each one to do his part of the work to the best of his ability, whether it be the turning of a spool or the putting on of a nut. It is not easy to determine what the final output should be in education, or what is the best method of securing the desired results.

The Aims of Education. The aims of education are numerous, and the methods of attaining the aims must vary with the child. Some prominent writers on education enumerate five or six aims.[1] A variety of individuals make necessary a variety of aims. Aimless teaching is wasteful teaching.

Educators are not agreed in regard to all of the ideals to be striven for in the teaching of arithmetic, and it is not probable that complete agreement will be reached upon this point. Teachers of history, geography, grammar, and the industrial arts are not in agreement in regard to all of the ideals to be attained in their respective subjects. Numerous reasons for the teaching of arithmetic have been advanced, but most people will agree, that, in the main, the subject has a two fold justification. It is taught because of its practical bread-and-butter value, and because of the opportunity that it affords for setting before the pupil certain standards in concise, logical thinking. Arithmetic is by no means the only subject that serves these two purposes, but that it does serve them in a distinctive way, few will deny.

No one questions the necessity of a mastery of certain

[1] See Bagley, *Educative Process*, pp. 40-65; O'Shea, *Education as Adjustment*, Chap. 4 and 5.

fundamental number relations. All admit that a child must acquire a certain mastery of quantitative relationships. It is not an easy matter to evaluate the mental training that a pupil gets from arithmetic that he does not get equally from other subjects. The pupil is probably more conscious in arithmetic than in any other subject in the grades that he has come into contact with certain truth. When the mathematician Laisant was asked the relative importance of the utility and the culture value of arithmetic, he replied that it is like asking which is the more important, sleeping or eating — the loss of either is fatal.

The teacher who recognizes but one of these aims is not teaching most effectively. "The practical side must concede to the disciplinary side by having its processes understood when they are presented, even though the child is not called upon to remember the reasoning. The disciplinary side must concede to the practical by selecting its topics in such a way as to give no false notions of business, and as to encourage the pupils to take an active interest in the quantitative side of the world about them."[1]

The Specific Aims of Arithmetic. The teacher should recognize not only the general purposes for which arithmetic is taught, but he should have clearly in mind the specific aims of the work in each of the grades in which he teaches. It is not wise to attempt to draw too fine a distinction between the aims of the work in each of the grades. These aims will be influenced more or less by local and individual factors, but, in general, certain purposes may be considered as of prime importance in the respective grades.

The object of the work of the first and second grades is to aid the pupils to image clearly the objects and groups of objects in proper number relations; to make clear and definite quantitative imagery. This may be accomplished

[1] Smith, *The Teaching of Arithmetic*, p. 21.

through emphasis upon counting, upon addition and subtraction, upon simple estimates and comparisons, upon games, rhymes, drawing, construction work, and such.

It is the specific object of the third and fourth school years to make the pupil proficient in addition, subtraction, multiplication, and division with whole numbers and certain fractions, as a basis for the work that is to follow. Emphasis should be placed upon accuracy and speed in both oral and written work. By the use of simple problems adapted to the experiences of the pupil, there should be developed the ability to interpret simple number relations and to reason from simple data. The pupil's appreciation of quantitative relationships in the life about him should be considerably developed in these grades, through continued use of measurement and estimates and a study of the commonly used tables.

In the fifth and sixth grades the mechanics of the fundamental processes with integers and fractions should be thoroughly mastered. There should also be increased emphasis upon the solution of problems. Considerable time should be devoted to the selection of appropriate processes for solving problems. Every effort should be made to develop independent thought by strengthening the judgment in the selection of correct processes. The use of appropriate checks should become habitual in these grades, and this will do much to increase confidence and to bring pride in accomplishment that is so desirable. Problems in these and all other grades should be, so far as possible, based on the pupil's experiences. The data of geography, history, science, and the household and industrial arts may be utilized to a considerable extent.

The pupil who passes the age of eleven or twelve without the ability to perform the fundamental operations with facility and a relatively high degree of accuracy is quite likely to be handicapped in these respects throughout life.

A great responsibility rests upon the teachers of the fourth, fifth, and sixth grades, and the pupil's success in arithmetic in the following grades depends to no small degree upon the results that are secured in these grades. Many a child is sent from these grades a cripple in his work in arithmetic. Attention to individual needs should be given, and much loss of time in later grades may thus be avoided.

It is the purpose of the arithmetic of the seventh and eighth grades to give a mastery of percentage and its applications, of mensuration, ratio and proportion, and of square root; to afford a thorough and comprehensive review of the entire subject of arithmetic; and to give the pupil a working knowledge of the graph, the formula, the equation, and of intuitive geometry. In these grades especially, the larger aspects of social, industrial, and economic life should be emphasized. The solution of numerous problems and the application of appropriate checks should be continued, and every effort should be made to develop independence of judgment and confidence in results.

The Elimination of Topics. The arithmetic of the recent past included several topics which should have been eliminated because they no longer served any practical end. Tradition has exercised a powerful influence in retaining topics in arithmetic after their period of usefulness has passed.

In former years, the defense offered for the retention of any topic that was no longer of practical value was that it possessed disciplinary value. Today we do not believe that any topic should be retained merely because of its disciplinary value. We seek the maximum of mental discipline in the topics that have some practical value. If a practical topic is properly presented, the disciplinary value will be obtained from it. Both insight and skill may be acquired from the study of such a topic.

Waste in arithmetic today is due to the fact that we

have not eliminated all topics that find no application in present-day practice. The established traditions of the schools, the tendency of teachers to teach as they were taught, and the fascination of some of the old types of problems, tend to perpetuate the obsolete. In progressive schools all topics in arithmetic are omitted which do not have a definite relation to real life, or which cannot be related to the pupil's experiences, or which are not clearly within his comprehension. It is wasteful to teach any topic in the grades that the pupil will not use in some way, either outside of the classroom or in the school work that comes later. Even in the latter case a topic should not usually be presented until some use is to be made of it.

Only the most elementary processes of arithmetic are commonly used by the majority of people in their daily activities.

What Topics Should Be Eliminated. The demands of the present insist that the topics enumerated below be omitted from a course in arithmetic in the grades.

1. Greatest common divisor and least common multiple of all numbers not readily factored, and all work involving the Euclidean method. The terms of common fractions today are relatively small and easily factored, or the fractions are expressed as decimals. Prior to the beginning of the seventeenth century, business practice demanded a knowledge of the long-division form of the greatest common divisor. It is probable that the subject of greatest common divisor and least common multiple will be omitted entirely from courses in arithmetic in a few years.

2. All obsolete tables in denominate numbers and all tables that are of use to the specialist only. Troy and apothecaries' weight are not of importance to most people.

3. All problems dealing with compound numbers of

more than two or three denominations and all reduction from troy to apothecaries' weight should be omitted.

4. All work in circulating decimals should be omitted; the topic should be studied as a part of infinite series in algebra.

5. All applications of percentage that do not conform to present-day practices should be omitted. True discount should not be taught. The numerous state rules on partial payment should receive no consideration unless they are in general use in the community. The types of negotiable paper should be those in common use among business men.

Equation of payments is not of practical value today to anyone except a few specialists in foreign commercial transactions. Annual interest should not receive emphasis.

6. Cube root should not be taught in the grades.

7. Progressions have no practical value in arithmetic. The theory of progressions is distinctly algebraic.

8. Compound proportion has been largely replaced by unitary analysis. Even simple proportion is of less importance than formerly, because of increased emphasis upon the simple equation and analysis.

9. Problems which require long and involved solutions or answers should be omitted. Those listed below, copied from an old arithmetic, (Musgrove and Wright's Commercial Arithmetic) are extreme examples of this.

Simplify

$$\frac{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}}{\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\frac{31}{32}+\frac{63}{64}+\frac{127}{128}}$$

Reduce the common fraction $\frac{1}{49}$ to a decimal.

Ans. .020408163265306122448979591836734693877551

Enriching the Course. We are attempting today to enrich our modified curriculum by introducing numerous problems from practical life. We are expecting to secure a better mastery of the fundamental operations through

greater emphasis on systematic, motivated drill and through greater emphasis on oral arithmetic.

The progressive arithmetic of today limits common fractions to those of everyday life. It emphasizes thrift and savings and safe types of investments. It emphasizes short methods, accuracy, and checks. It makes prominent the social aspects of arithmetic by emphasis on practical problems of the home, the store, the farm, the bank, the shop, and the mine, the minimum essentials of arithmetical needs.

Unnecessary Computation. There is waste of time and of effort in arithmetic whenever a process is carried further than the data upon which it is based would justify. No result in mathematics can be more accurate than the data upon which it is based. Many pupils never appreciate the absurdity of finding the interest on a given sum to within a millionth part of a cent, or of finding the circumference of a carriage wheel to within a millionth of an inch. For most practical computations a result computed to two or three decimal places is sufficiently accurate.

Time is wasted because of the performance of unnecessary operations. It is wise never to multiply until you are forced to, and never divide until you are obliged to. The following problem will illustrate this point.

Required to find the radius of a circle equal to the combined area of two circles of radii 6 and 8 inches, respectively.

NECESSARY WORK

The area of the required circle $= 36\pi + 64\pi = 100\pi$

The radius of any circle $= \sqrt{\frac{a}{\pi}}$

The radius of the required circle $= \sqrt{\frac{100\pi}{\pi}} = 10.$

In solving such a problem, many pupils would multiply

both the 36 and 64 by the numerical value of π, then add the products. This multiplication is unnecessary.

Cancellation frequently enables one to save time in the solution of a problem. For example: If 29 bushels of potatoes sell for \$20.88, what will 31 bushels sell for at the same rate?

Solution

$$\frac{31 \times \$20.88}{29} = \$22.32$$

Time Devoted to Arithmetic. For many years arithmetic was the foremost study in the curriculum in determining the pupil's standing. It is still an important subject in this respect. Pupils formerly devoted about one-third of their school time to the study of arithmetic. As early as 1850, other subjects which were demanding admission to the curriculum sought to curtail the time given to arithmetic. Today fifty of the leading cities of the United States devote an average of 15.2 per cent of the school time to the subject. The decline in the time devoted to arithmetic is not due to a feeling that arithmetic is less worthy, but to the demands of other subjects for admission. A sufficient number of school hours is still devoted to the study of arithmetic. What the schools need is not more time for the subject, but a clearer realization of the purposes for which arithmetic should be taught, and a better adjustment of means to ends. We need to economize the time that is now devoted to the subject rather than waste it by aimless and unsystematic instruction.

Applications of Arithmetic. It is wasteful to teach arithmetic without sufficiently emphasizing the applications of the subject to life. The study should help the pupil to interpret life from the quantitative point of view, and unless it does this the subject is not fulfilling its purpose. A pupil should be taught arithmetic so that he can apply

it to problems outside of the book. Many pupils can solve complicated problems about prisms, pyramids, and cones, but cannot find the capacity of their father's coal bin or the amount of dirt that must be removed in digging a cellar. Unless the facts of the classroom are interpreted in terms of facts outside the classroom the teaching of the subject is not efficient. Many of the experiences of the pupil in the school should vitalize the experience outside of the school, and many of the experiences outside of the school should be utilized to clarify the work of the school. Pupils are too often required to solve problems of whose use and application they can have no clear conception.

Unity of Arithmetic. It is wasteful to teach arithmetic as a multitude of unrelated topics when in reality the subject contains only a few distinct processes. The pupil should appreciate the fact that the same process may appear under various phases of the subject. One who has no appreciation of the unity and the simplicity of arithmetic has failed to grasp the full significance of the subject; arithmetic to him is a maze of rules and processes. The teacher should show the pupil how, from a few definitions and fundamental processes, the entire science is developed step by step. To present a new topic without showing its relation to those that have preceded is wasteful teaching.

The Thought Side of Arithmetic. There is waste in the teaching of arithmetic because not enough attention is given to the development of the pupil's power to reason. Not enough training is given in the selecting of the proper process to be used in a given situation.

Mechanical processes are of great importance in arithmetic, and our pupils have not become too expert in them; but in many schools mechanical ability has been developed largely at the expense of the ability to see relations and to think, to select appropriate processes with certainty. Frequently too small a proportion of the arithmetic period is

given to developing the thought side of the subject. The pupil should be able to perform the fundamental operations with integers and fractions with reasonable facility and with a high degree of accuracy; but it is necessary also that he should know when the performance of a given process is necessary in the solution of a given problem. If the judgment of the pupil is to be developed in arithmetic, the teacher should exercise great care that he does not hamper it by the imposition of numerous rules and involved explanations.

Making of Assignments. There is much waste of time and of effort in the study of arithmetic because the teacher does not make the assignment with definiteness and precision, and as a result of this the study hour is one of blind groping and discouraging failure, resulting in disgust for the study and in dawdling habits. All assignments are intended to afford growth toward independence and initiative in thought and action. The immediate purpose of an assignment varies, but in general its purpose is to guide the pupil in preparing for whatever the new recitation will present, to direct him toward the accomplishment of what he can do by himself, and to save time in the recitation proper.

The teacher should always have a definite and satisfactory reason for assigning each lesson. This reason should be based on the teacher's knowledge of the pupils, their interests and needs, and on a knowledge of the lesson. A well-made assignment saves time in succeeding recitations, and makes the recitation more profitable than it would possibly be without the preparation of the child's mind that is afforded through the assignment.

All assignments should be made with great care and precision. As the pupils grow in power they may sometimes be permitted to suggest the assignment, which will be discussed and amended by the teachers. Some classes,

late in the grades, may occasionally be trusted to make the assignment. This affords an excellent means of strengthening the selective power and of encouraging originality. A good assignment will put the pupil in a mood to work on his lesson and will stimulate him to attack and solve the problems before him.

Explanations by the Teacher. It is questionable how much explanation should be given by the teacher when the assignment is made. Some assert that the teacher should not point out the difficulties, but should leave the child free and unhampered by any assistance. This view will probably entail a great waste of time and energy in most classes. It often leads to the magnifying of trivialities and results in unorganized knowledge.

It is probably good practice in the lower grades for the teacher to explain nearly all the difficulties, and in the upper grades to make the assignment in such a way as to assist in removing them. One measure of the value of an assignment is the degree of interest aroused and maintained. If the teacher knows both his pupils and the subject to be taught, he should be able to arouse and maintain the interest of the pupils in the subject.

If in the recitation, or even when making the assignment, the teacher works up to an absorbing point and then leaves it in suspense, the pupils will return to the subject with keen interest during the study hour.

Enough time should be devoted to the assignment to make it clear and definite — to remove the obscurities and certain difficulties, and to create enough interest to insure further study. Sometimes this can be done in a few minutes; at other times it will require most of the recitation period. The assignment is a place for raising problems and creating interest in those problems. Haste and slovenliness are to be avoided because they are destructive of energetic effort

on the part of the pupils. No general directions can be given as to the proper time to make an assignment. Usually it should be made at the beginning of the recitation period. If the assigning of the advance lesson is postponed until the close of the period, teachers frequently find that they have not allowed themselves sufficient time, and the result is a hurried assignment, lacking both definiteness and precision.

A teacher has no right to expect a pupil to prepare with precision an assignment that was not made with care and with definiteness. It sometimes happens that the advance assignment is to depend largely upon points to be developed during the recitation period or upon the extent to which the recitation has been satisfactory. Under such conditions the teacher cannot know the appropriate length for the assignment or what features should be made most prominent until late in the period. Some excellent teachers always make the assignment at the close of the period, but the custom is not one that can be generally practiced with the best results. Individual assignments, such as telling a pupil that he will be held responsible for the solution and explanation of a given problem at the next recitation, may properly be made during the recitation period.

Methods of Study. One of the greatest sources of waste in education is found in faulty methods of study. Many earnest pupils who are anxious to study do not know how to do so independently, intelligently, or economically. The teacher should do all that he can to assist the pupil to form good habits of study.

Pupils should learn to overcome difficulties without the aid of the teacher. A spirit of self-reliance and of tenacity of purpose are excellent assets for any pupil. Interest in one's work may be initiated and augmented by consistent application to a task that may originally have been without interest. Many a pupil has become tremendously interested

in his mathematics by persevering over tasks that were at first uninteresting to him.

If a pupil is to prepare his mathematics assignment with economy of time and of effort, the conditions under which he studies should be such as to admit of a high degree of concentration upon the work in hand. The habit of concentrating the attention is of great value in studying any subject, and especially mathematics. Many pupils do not realize the necessity of selecting, when this is possible, an environment that is conducive to study, and that will favor concentration upon the task. If parents and pupils can be convinced that under proper conditions twice the work may be accomplished in half the time, a great gain has been made.

Waste in Mechanical Procedure. There is waste of the recitation period because the teacher permits pupils to form dawdling habits of work, especially at the blackboard. This can usually be avoided by attention on the part of the teacher. In many classes much time is wasted in such procedures as the calling of the class roll, passing to the blackboard, and the distribution of papers. The writer visited a class in arithmetic in which the teacher consumed eight of the forty minutes allotted for the recitation in the calling of the roll and the distribution of papers. During this time no comments or suggestions were made in regard to any of the work. This teacher wasted about 20 per cent of the recitation period each day in such ways as have been indicated.

As the teacher is, the class tends to be. A teacher who does not work in class with economy of time and of effort will soon find that many of the pupils have developed habits of work quite similar to his own. If a teacher wishes the maximum amount of work to be accomplished during the recitation period, even the minute details of the work must be planned with this end in view. Not every pupil in a

class will adjust himself to the ideals of the teacher, but it is certain that unless the teacher himself has proper standards of work the general average of the class in this respect will be materially lowered.

Reviews and Examinations. A few persons advocate the elimination of reviews and examinations from school work. They say that such eliminations will delight the American boy and that no compulsion will be necessary to induce him to attend school. School life will be a continuous round of pleasure, and there will be no work to make Jack a dull boy. Such proposals are without social or psychological warrant. Sociology informs us that it is the part of wisdom to retain those things that long and successful experience have proved to be worthwhile. Psychology informs us that drills and reviews are necessary in the fixing of habits and in organizing material.

It is a principle of psychology that if one is equally hazy about two things formerly learned, one months ago, and the other recently, it requires a smaller amount of effort to recall and fix the older of the two than it does to recall and fix the one more recently acquired.

Reviews properly distributed pay tremendous dividends in the economy of mental life. Surely the old practice, which provided for the automatic mastery of the fundamentals, should not be discontinued without a hearing. Many things still need to be reduced to the automatic or to be firmly fixed in the mind by means of drills and reviews. No true scholastic attainment or efficient education is possible without a reasonable use of reviews and drills.

Many pupils are in more danger of suffering from intellectual starvation than from mental dyspepsia. A school is organized and maintained in order that it may be instructed, and reviews are an essential part of efficient instruction. The social or educational reformer who would

eliminate drills and reviews is striking at one of the essentials of instruction.

In individual lessons attention is focused primarily upon the various parts of the subject matter rather than upon the unity of the parts. A good review transfers the attention to the larger relationships. The unity of a subject is likely to be lost in the multitude of details, and a comprehensive and thorough review is necessary, not only to fix the important points more firmly in the mind, but to produce that thoroughly organized and well-articulated knowledge which is such a valuable asset.

The ingenuity and skill of the teacher are frequently taxed to the utmost in order to devise ways and means of instilling a high degree of interest into a review lesson. It is not necessary that the subject matter and the method of presentation be the same as when the topics were previously studied. Indeed, the advance that has been made will frequently suggest a broader view and a better method.

Thorough and comprehensive reviews are especially necessary at the close of the year's work. Investigations on this point indicate that such reviews are economical in that they save time in the work of the succeeding year. A brief review of the principal features of the previous year's work should also be given during the early part of the school year.

It is the custom today in some localities to decry examinations and to urge that they be eliminated. It is doubtless true that many examinations in the past were given with little or no thought as to their real purposes.

Examinations should usually be comprehensive in character; their purpose should be to aid the pupils in organizing and correlating the knowledge acquired into a coherent and systematic unit. The examination should be regarded as a means to an end, and not as an end in itself. Too often the examination is regarded simply as a means of testing

the memory for specific and unrelated facts. It is proper that a part of an examination should be devoted to this purpose, but the ability to be tested should be the ability to organize and to apply.

Examinations given by the teacher, if well planned and wisely constructed, frequently aid the teacher to discover individual or class weaknesses, and to furnish a basis for further instruction. Such examinations, if given with due regard for the welfare of the pupils, are justifiable.

Examinations given by superintendents, principals, or supervisors usually have as their aim the setting of standards of work throughout the system, the testing of the efficiency of the course of study, and the discovery of individual weakness. Such examinations, if properly conducted, may be made an effective educational aid. Broad-minded teachers usually welcome the opportunity to have their work tested.

The present tendency is to decrease the frequency of examinations and to use more care in their preparation. The marks received on examinations formerly determined whether a pupil was to be promoted. This is not true in many schools today. The practice of giving "finals" is now uncommon. The old type of examination is being replaced by those which require old knowledge to be utilized in new situations. More examination questions demanding thought, judgment, and choice are given today than in former years.

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See also *Suggested Readings,* page 68.

CHAPTER VIII

ACCURACY

Prevalence of Inaccuracy. Many critics of the school of today assert that the pupils have become lax and careless in thought and in expression. The modern pupil studies many things which were not taught in the schools of the last generation, and there is some truth in the assertion that a part of this knowledge is superficial and inaccurate, "a collection of vague ideas rather than clear cut notions about definite things." It is asserted that while the pupils of today can think and write on more subjects than could the students of former years, their expressions are less clear and coherent.

Teachers should not turn a deaf ear to the complaints about inaccuracy of thought and statement. The complaints come from many sources. The employer maintains that it is difficult to hire a boy who is accurate and who has a mastery of even the four fundamental operations. Many teachers in the upper grades state that the pupils who enter their classes are not well prepared to carry the work, because their thinking is illogical and their ability in computation is poor. Thorndike says "it is much better for a child to know half of the multiplication tables, and to know that he does not know the rest, than to half know them all; and this holds good of all the elementary bonds required for computation."

Any bond should be made to work perfectly even though slowly, very soon after its formation is begun. Speed can easily be attained by proper practice.

One writer asserts that all the complaints about inaccur-

acy are an evidence that accuracy is one of the goals which education ought to reach. Ex-President Eliot enumerates among the essentials of the cultured mind the ability to think clear and straight. Accuracy is one of the marks of the scholar. Any system of education which minimizes the importance of accuracy of thought and of expression relegates to a subordinate position one of the essentials of true scholarship and culture.

Accuracy of Thought. If there is one subject rather than another in the curriculum which should be characterized by a high degree of accuracy, that subject is mathematics. In mathematics a statement is either right or it is wrong; there is no middle ground. Mathematical accuracy is proverbial. The demand of the present is for greater accuracy in both thought and computation. Accuracy of thought is the more fundamental and is quite frequently the basis for accuracy of manipulation. Myers defines accuracy of thought in arithmetic as "the degree of closeness of expression to idea — the adequacy of assertion to thought." As the pupil matures, his accuracy of expression should improve, as measured by an absolute standard, for his conception of quantitative relationship is extended. The expression used by the pupil often exposes the mental process.

A great responsibility rests upon the teachers of the early grades for securing accuracy of expression in oral and in written work. In these grades inaccurate habits of expression are often formed and if not corrected before the pupil reaches the upper grammar grades they are quite likely to persist and to interfere seriously with the pupil's progress in the subject. David Eugene Smith says, "It is the loose manner of writing out solutions, tolerated by many teachers, that gives rise to half the mistakes in reasoning which vitiate the pupil's work," and "teachers are coming to recognize that inaccuracies of statement

tend to beget inaccuracy of thought and so should not be tolerated in the schoolroom." There is real value in clear cut and concise statements of conditions and arguments. One who has observed recitations in a large number of schools is impressed by the looseness and carelessness of expression that is frequent in both oral and written arithmetic.

A number of the more common inaccurate statements found in certain textbooks and frequently permitted to go unchallenged in classrooms will now be considered. The fact that certain of the inaccuracies considered are found in textbooks of merit does not lessen the justness of the criticisms that follow. A statement is not correct merely because it is used by a large number of authors. The question at issue is not whether this or that author has a correct statement for the idea under consideration. "Not who is right, but what is true," is the question at issue. The good teacher will be open-minded, and will consider the question in an impersonal way.

If enough people of influence habitually use an inaccurate statement until its "real" meaning becomes apparent it is hardly worth while to raise objection to the inaccuracy. The inaccuracies which follow are not of this type.

Figures and Numbers. One common inaccuracy is due to a confusion between the symbol and that which the symbol represents. Frequently the pupil is directed to add figures. Figures are not numbers; they are symbols which represent numbers. It is possible to perform various operations with numbers, but these operations cannot be performed upon that which merely represents numbers. It is important that teachers and pupils discriminate between numbers and the symbols which represent them. The teacher is reminded that accuracy is not a little thing, though we should be accurate even in little things.

Inaccuracy in Statement. Such statements as the follow-

ing are frequently found in classroom work and in some textbooks:

$$3+4=7+5=12\times2=24$$

The inaccuracies in the above statement are evident. It is not true that $3+4=7+5$, nor does $7+5=12\times2$. The mere fact that the correct answer is obtained does not justify the inaccuracy of statement. If the data given are correctly used and the computations are accurate, the answer for any problem will be correct; but it is possible to obtain the correct answer to a problem, even though no step in the solution is correct. This may be brought about by a balancing of inaccuracies in computation. The process may justify the answer, but the answer does not justify the process.

If the above example needs solution, it should be put into the following form: $3+4=7$; $7+5=12$; $12\times2=24$.

Inaccuracy in the Addition of Mixed Numbers. A third inaccuracy is the one shown below to the left.

$4\frac{2}{3}=\frac{8}{12}$
$3\frac{1}{2}=\frac{6}{12}$
$2\frac{3}{4}=\frac{9}{12}$
$9\ +\frac{23}{12}=10\frac{11}{12}$

It is apparent that $4\frac{2}{3}$ does not equal $\frac{8}{12}$ as the solution asserts, nor does $3\frac{1}{2}$ equal $\frac{6}{12}$ or $2\frac{3}{4}$ equal $\frac{9}{12}$.

The solution may be correctly expressed in several ways. The one to the right does not violate the truth by any of its statements.

$4\frac{2}{3}=4\frac{8}{12}$
$3\frac{1}{2}=3\frac{6}{12}$
$2\frac{3}{4}=2\frac{9}{12}$
$10\frac{11}{12}$

Inaccuracies in Multiplication and Division. In multiplication the multiplier must always be abstract, and the product must be of the same denomination as the multiplicand.

In division the quotient is always of the same name as the dividend when the divisor is abstract.

Such inaccurate statements as the following are not uncommonly made in classrooms:

$2 \times \$50 = 100$	$\$100 \div 4 = 25$
$2 \times 50 = \$100$	$100 \div 4 = \$25$
	$\$100 \div \$4 = \$25$

Feet Multiplied by Feet. The following is an inaccurate expression: 4 ft. $\times$ 5 ft. = 20 sq. ft. The first factor (4 ft.) cannot represent the multiplier, because it is not abstract; for the same reason the second factor (5 ft.) cannot be the multiplier. Moreover, since the product must be of the same name or denomination as the multiplicand, it is apparent that if the result (20 sq. ft.) is correct, neither 4 ft. nor 5 ft. can be the multiplicand, for they are not of the same name as the product. This multiplying of *feet* by *feet* and calling the result *square feet* is one of the common inaccuracies of statement in arithmetic. If a rectangle is 4 ft. wide and 5 ft. long, we may find its area by the following method: 4×5 sq. ft. = 20 sq. ft.

Common Fractions. In the analysis of problems in fractions and in percentage, inaccuracies of expression are common. The following example will indicate the nature of some of these inaccuracies:

Problem. Two-fifths of a number equals 12. Find the number.

I. Incorrect Solution	II. Correct Solution:
$\frac{5}{5}$ = the number	$\frac{5}{5}$ of the number = the number
$\frac{2}{5} = 12$	$\frac{2}{5}$ of the number $= 12$
$\frac{1}{5} = \frac{1}{2}$ of $12 = 6$	$\frac{1}{5}$ of the number $= \frac{1}{2}$ of $12 = 6$
$\frac{5}{5} = 5 \times 6 = 30$	$\frac{5}{5}$ of the number $= 5 \times 6 = 30$

Not a statement in Solution I is true. If the number is equal to $\frac{5}{5}$, it must be true that the number is equal to 1, since $\frac{5}{5}$ is equal to 1; but the number is 30 and not 1. If $\frac{2}{5}$ is equal to 12, then $\frac{2}{5}$ must be equal to $\frac{60}{5}$, the equivalent of 12; but this is evidently absurd. If $\frac{2}{5}$ is equal to 12, then $\frac{1}{5}$ must be equal to 6; but 6 is equal to $\frac{30}{5}$, and $\frac{1}{5}$ does not equal $\frac{30}{5}$. The last statement of Solution I is seen to be absurd when we substitute 1 for its equal, $\frac{5}{5}$, for we then have $1 = 30$. To say, "Let $\frac{5}{5}$ equal the number" or "Assume that $\frac{5}{5}$ equals the number," does not eliminate the inaccuracy. The incorrectness of Solution I is not an incorrectness in reasoning after the first steps are taken. The reasoning is faultless after the first two steps. The incorrectness lies in the inaccuracy of statement. The first two statements are untrue, hence the reasoning based upon them is untrue. An absurd conclusion may be reached by correctly reasoning from false premises.

At first sight it may appear that Solution II does not differ greatly from Solution I, but a brief study of the two solutions will convince one of the inaccuracy of the first and the correctness of the second. The word "of" when used with common fractions indicates multiplication. Since $\frac{5}{5}$ is a common fraction, the first statement of Solution II is equivalent to "1 times the number equals the number." This is always true. Contrast this with the corresponding step of Solution I, which asserts that $\frac{5}{5}$ (or 1) equals the number. This last statement is never true, unless the required number is actually 1. The second statement in Solution II is given in the original problem, and hence is known to be true. From these two statements, or premises, both of which are true, by accurate reasoning we arrive at the conclusion which is true.

Many teachers may feel that their pupils do not have time when analyzing problems of this type to write an analysis similar to Solution II. It need only be urged that

no teacher and no pupil should ever be so pressed for time or for space that he hasn't time to tell the truth. It is not correct to say that $\frac{5}{8}$ equals A's age, or A's money, or the distance from B to C.

It is not intended in the preceding examples to convey the idea that the suggested forms are the only correct forms. A statement of facts may often be made in a variety of ways, and it matters little how the fact is stated, provided only that the statement is accurate, concise, and clear.

Inaccuracies in Percentage. The analysis of problems in percentage often leads to inaccuracies similar to the one just considered.

"Forty per cent of a number is equal to 80. Find the number." If we begin the solution by asserting that 100% equals the number, we are evidently using the same inaccuracy of statement that was pointed out above, since 100% equals $\frac{100}{100}$, and it is not correct to say that $\frac{100}{100}$ equals the number. The correct statement is 100% of the number equals the number.

Mensuration. It is incorrect to speak of the square root of 49 square feet as 7 feet. By the *square root* of a number is meant that which multiplied by itself produces the given number. Thus, the square root of 25 is 5, because $5 \times 5 = 25$. If the square root of 49 sq. ft. = 7 ft., it follows that 7 ft. $\times$ 7 ft. = 49 sq. ft.; but from the argument immediately preceding, it appears that this is impossible. It is incorrect to say 27 cu. ft. $\div$ 9 sq. ft. = 3 ft., since 9 sq. ft. $\times$ 3 ft. does not equal 27 cu. ft. For justification of this point and for fuller explanation, see the chapter on *Mensuration.*

The preceding inaccuracies are common in our schools, and they are in direct contradiction to the habits of accuracy of expression which should be cultivated in the study of arithmetic. Moreover, such inaccuracies as those just pointed out are likely to lead to confusion and to inaccuracy

of thought. Teachers should not tolerate them in their classes. Such inaccuracies of expression cause "many teachers to sit up at night to correct mistakes which they had better sit up in the daytime to prevent."

Inaccuracy in Computation. Not only should accuracy of statement be secured, but a high degree of accuracy of computation should be acquired. The evidences of inaccurate computation in our schools are overwhelming. It is not unusual to find a group of seniors in high school, not 25 per cent of whom can add ten numbers of four digits each and secure the correct result the first time.

Many business men contend that it is very difficult to secure a boy who can add, subtract, multiply, and divide with speed and with a high degree of accuracy.

When a boy or girl seeks a position in which accuracy and rapidity of computation are essential, it is of little consequence that the applicant can solve problems and explain them in a clear and concise manner, if the applicant does not have the ability to perform the arithmetical operations with speed and accuracy. In many vocations work that is inaccurate is of no value.

Accuracy in Practical Work. It is not possible to secure absolute accuracy in computations which involve measurements. The desired degree of accuracy depends on the purpose to be served by the computation. The Coast and Geodetic Survey uses an ice-bar apparatus for measuring a base line. An ordinary metal bar would be subject to expansion, and the results would not be sufficiently accurate. By supporting the metal measuring-bar in a trough packed in ice, the bar is maintained at a uniform temperature, and a base line can be measured with an error of only 1 part in 2.5 million. In 1900 the United States Coast and Geodetic Survey completed the measurement of an arc along the 39th parallel from Cape May in New Jersey to Point Arenas in California, a distance of 2,625 miles. So

carefully was the work done that the total amount of probable error does not exceed 100 feet.

Many pupils go through the entire school course without appreciating the value of accuracy. Arithmetic as a tool is almost useless unless it has an edge keen enough to do its work with considerable speed and accuracy. Teachers permit and sometimes encourage inaccuracy by giving high grades for the correct process, even though the result is absurd. Pupils will more quickly form the habit of accuracy when they find that inaccurate results are marked zero. The good is the enemy of the best, in the arithmetic class as well as in many other places. It is not easy to secure work that is characterized by a high degree of accuracy. The teacher must continually insist upon accuracy, but the results justify the time spent and the effort expended. Mere practice will never make the shiftless and careless pupil accurate in computation. We must not permit our pupils to regard accuracy of computation as a matter of small importance. Some of the pupils of our schools seem to believe that accuracy in computation is of so little moment as to be almost beneath their intellectual dignity. It is hard to eradicate such an idea after the pupil reaches high-school age.

Speed and Accuracy. It was formerly asserted that the most rapid computers are always the most accurate. The relation is positive, but is not absolute. In Stone's investigation the school system which ranked first in accuracy was fourth in rapidity, while the system which ranked second in accuracy was tenth in rapidity. The effect of drill on speed and upon accuracy, as indicated by some investigations, is considered in the chapter on *Drill.*

"The common notion that speed is opposed to accuracy is correct when it means that the same person will tend to make more errors if he works at too rapid a rate; but it is entirely wrong when it means that the kind of person who

works more rapidly than the average person is likely to be less accurate than the average person."[1]

The subject of checks is closely related to that of accuracy of computation. Even the mathematician does not guarantee the accuracy of his results until he has applied some adequate check. Some simple checks for accuracy of results should be extensively used in the schools, and a number of such checks are considered in the following chapter.

Copying Figures. Copying figures is one of the tests in certain civil service examinations, and, because of its importance, it should have some consideration. One of the authors investigated a group of seventh-grade pupils with respect to inaccuracy in copying figures from a printed page. Mistakes of this type are of three general kinds; first, the inserting of figures that do not occur in the original copy; second, the omission of figures; and the third, the interchange of figures which actually do occur.

Pupils who are below the average in speed mis-copy figures more frequently than those who are above the average. Figures were omitted in the copying about one-third as frequently as they were mis-copied.

In a long test in the seventh grade involving only addition, subtraction, multiplication, and division, it is probable that from 4 to 8 per cent of all mistakes are due to mis-copying or omission of figures. No doubt this percentage is higher in the lower grades.

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[1] Thorndike — *The Psychology of Arithmetic,* p. 300.

CHAPTER IX

CHECKS AND THEIR USE

In the preceding chapter the necessity and importance of accuracy of thought and expression were considered. The effect of drill upon accuracy of computation is considered in the chapter on *Drill*. Another method by which increased accuracy of computation may be secured will now be considered.

The Necessity of Using a Check. Long practice in mathematical computations is indispensable to both accuracy and speed, but practice alone is not sufficient to produce the degree of accuracy which is essential. Even accountants and mathematicians are not willing to guarantee the accuracy of their computations until an appropriate check has been applied. Computation which is not characterized by a relatively high standard of accuracy is of little value.

One of the best ways to encourage pupils to think is to have them verify their results. The pupil who checks an arithmetic solution must go over his work again with care. Guessing by pupils may be restrained by requiring them to verify their work.

The navigator who computes his position at sea must check his results if he would guard the lives and the cargo entrusted to his care. A mistake of a mile in computing his results may mean the loss of hundreds of lives and of thousands of dollars. If a result will check, it is correct; if it will not check, it is incorrect. The best check to be applied is determined by a number of considerations.

The Teacher as a Check. Checks are of many kinds. The teacher may serve as a valuable check upon the work

of the pupil, but many teachers, by habitually accepting inaccurate and untidy work, do not stimulate the pupil to greater accuracy. Frequently work that is not correct in every particular should be considered wholly wrong and the teacher who habitually accepts work that is not accurate is unconsciously discounting the value of accuracy. The teacher should always be a check upon the work of the pupil, but the check of greatest value to the pupil is the one that he himself applies.

Pupils Should Check Results. Unless a pupil has acquired the habit of checking results, he is not master of the situation. Answer books are of value chiefly in checking unusually long or involved problems. The earlier the constant verification of results is begun, the more automatically it will be practiced. It is not uncommon for teachers to object to the use of checks on the ground that they diminish the number of problems that it is possible for a pupil to work. The fact that the check itself is a problem is frequently overlooked. The essential thing is to gain well-grounded self-confidence; to know that the result is correct.

The checks discussed in the following pages are commonly used by those in the various vocations in which accurate computation plays an important part. Not all of the checks should be taught to any class. One good check, for a given process, so well known and so frequently used that its application has become almost automatic to the pupil, is of more value to him than three or four checks all of which can be used but no one of which has been thoroughly mastered. Many teachers make the mistake of familiarizing their pupils with several checks but not requiring them to use any one of the checks long enough to make it really practical.

Checks for Addition. Checking addition by combining the addends in the reverse order is so well known as to

render comment unnecessary. To combine in the same order a second time is no guarantee of the correctness of the result. The mind tends to repeat its own errors and if two numbers are incorrectly combined the first time, the chances are that the same error will be made if the order of the addends is not changed.

Add:	(*a*)	(*b*)
4823		
6792	23	34
8437	22	25
9265	25	22
7426	34	23
36743	36743	36743

The sum for the first column is 23. The sum for the second column, without carrying, is 22; for the third column the sum is 25, and for the fourth it is 34. These sums may be written as under (*a*) or (*b*) and then combined. The above methods are frequently used by persons who are likely to be interrupted while adding.

Casting Out Nines. Another check for addition is that of casting out the 9's. The following example will make clear the method of applying this check. Suppose we wished to check the result, 36,743, in the example cited above. Add the numbers in each of the addends and divide the sum by 9. The remainder is called the "excess." It is the remainder which would be found if the entire number were divided by 9. After the excess of 9's in each addend has been ascertained, add these excesses and divide this result by 9. The remainder (or excess) thus obtained should equal the excess in the sum in the original example.

	Excesses
4823	8
6792	6
8437	4
9265	4
7426	1
36743	23 = sum of the excesses.

The excess in the sum, 36,743, is 5. This is also the excess in the sum of the excesses.

RULE. The excess in the sum must equal the excess in the sum of the excesses.

Another example is given without explanation.

	Excesses
9426	3
3875	5
2873	2
6941	2
2739	3
25854	15 = sum of the excesses.

The excess in the sum, 25,854, is 6. This is also the excess in the sum of the excesses.

After the check has been applied several times, the pupil should find it unnecessary to write down the various excesses. After a brief period of drill, the entire check can be applied without writing down any of the figures. Moreover, the pupil will soon discover that it is wise to group the digits of a number into combinations whose sum is 9, and that the grouping enables one to determine the excess in a number very quickly. For example, it is desired to find the excess of nines in 32,789. The practiced eye will

group the 7 and the 2, and will see that the only thing to do thereafter is to determine the excess of nines in 8+3. Instead of adding the digits and then dividing the sum by 9, pupils should accustom themselves to add the digits until the sum is 9 or more, then add only the excess above 9.

It is apparent that the check by casting out the 9's is not a proof of the accuracy of the result. If two digits in the result were interchanged, or if the errors were of such magnitudes as to balance each other, the check would not detect the errors. Moreover, the check is not short, and it is more liable to error than a new addition, unless as much time is given to drill on casting out of 9's as to the addition itself, and this is not desirable.

Checks for Subtraction. Subtraction may be checked by casting out the 9's, but the well-known checks of adding the difference to the subtrahend to produce the minuend, or of subtracting the difference from the minuend to produce the subtrahend, are more easily applied.

Checks for Multiplication. Multiplication may be checked in a number of ways. Probably the methods most familiar to the majority of teachers are the dividing of the product by the multiplicand to produce the multiplier, or dividing the product by the multiplier to produce the multiplicand. The check by casting out of the 9's is especially adapted to multiplication and is easily applied. The following will exemplify the method of applying this check.

	Excess
4836	3 (excess of 9's in 4836)
285	6 (excess of 9's in 285)
24180	
38688	
9672	
1378260	18 = product of the excesses

The excess in the product, 1,378,260, is 0, which is also the excess in the product of the excesses.

RULE. The excess of 9's in the product must equal the excess in the product of the excesses of the multiplicand and the multiplier.

A second example follows:

	Excess
3725	8 (excess of 9's in 3725)
439	7 (excess of 9's in 439)
33525	
11175	
14900	
1635275	56 = product of the excesses

The excess in the product, 1,635,275, is 2, which is also the excess in the product of the excesses.

Some multiplications may be conveniently checked by interchanging multiplicand and multiplier. An example will make this clear.

427	326
326	427
2562	2282
854	652
1281	1304
139202	139202

Checking multiplication by the above method becomes inconvenient in such cases as 47839×37.

Checks for Division. Division may be checked by determining whether the result obtained by multiplying the quotient by the divisor and adding the remainder, if any, to this product, equals the dividend. Casting out the 9's may be used to advantage in checking division. The first

example that follows shows the method of checking division by casting out the 9's when there is no remainder. The second shows the method when there is a remainder.

Divide 1635275 by 439.

```
    3725
439)1635275
   1317
   3182
   3073
   1097
    878
    2195
    2195
```

The excess of 9's in the dividend, 1,635,275, is 2.

The excess of 9's in the divisor, 439, is 7.

The excess of 9's in the quotient, 3725, is 8.

The product of the excesses in the divisor and quotient is 56, and the excess in 56 is 2. This is also the excess in the dividend.

RULE. When there is no remainder, the excess in the dividend equals the excess in the product of the excesses of divisor and quotient.

```
     67
847)57426
    5082
    6606
    5929
     677
```

When there is a remainder, the check is applied as follows:

The excess in the dividend is 6; in the divisor it is 1; in the quotient it is 4; and in the remainder, 2.

The product of the excesses in the divisor and quotient (1×4) is 4. Add the excess, 2, in the remainder; the sum is 6. Cast the 9's out of this sum if it is greater than 9. The result, if correct, must equal the excess in the dividend. Practice will enable one to use these checks with facility.

Approximate Checks. A rough or approximate estimate of the result before a computation is made is frequently a valuable check. Too often pupils obtain an answer of

$480 when the result should have been $4.80, or a result of $54.16 when the correct result is a hundred times as large. If teachers would drill their pupils more frequently in approximating results, the number of absurd answers would be reduced. If a problem requires the cost of 520 yards of cloth at 13 cents a yard, the pupil should know whether the result is about $6.50, $65, $650, or $6,500. If the problem requires the cost of 432 articles at 34 cents each, the pupil should see that since the cost is about $\$\frac{1}{3}$ each, the entire cost will be about one-third of $432, or $144.

Self-Checking Problems. Self-checking examples and problems serve a useful purpose, but checking should not be confined to this type. The problem shown to the left will exemplify this type.

$$\begin{array}{r} 3+\ 5+\ 7+\ 6=21 \\ 2+\ 8+\ 9+\ 3=22 \\ 5+\ 4+\ 8+\ 9=26 \\ 7+\ 2+\ 6+\ 5=20 \\ \hline 17+19+30+23=89 \end{array}$$

The admissions to a certain ball park were as follows:

	Wed.	Thurs.	Fri.	Sat.	Totals
Children.....	428	369	514	483	1794
Women.....	387	563	618	327	1895
Men.....	942	1048	1127	1026	4143
	1757	1980	2259	1836	7832

$$\begin{array}{r} 6\times\ 38=\ 228 \\ 6\times\ 29=\ 174 \\ 6\times 126=\ 756 \\ \hline 6\times 193=1158 \end{array} \qquad \begin{array}{r} 336\div 4=\ 84 \\ 192\div 4=\ 48 \\ 284\div 4=\ 71 \\ \hline 812\div 4=203 \end{array}$$

Numerous self-checking problems based on local conditions will suggest themselves to most teachers. The election returns, the enrollment in various grades of the school, the number of pieces of mail of various classes

handled by the local postoffice in a month, are among the problems that may be used.

Occasional checking is of only slight value. The pupil should get correct results and should know that the results are correct. Habitual checking tends to beget confidence in one's results, and confidence in one's ability to produce correct results in a given field lies near the basis of success in that field. The pupil who lacks confidence in himself may actually retrograde in his work, because lack of confidence will undermine his power of action.

CHAPTER X

THE NATURE OF PROBLEMS

The Changed Character of Problems. Textbook makers are limited in the selection and organization of their material by two facts: First, changing social conditions, and second, the strengths and limitations of the different maturity levels of childhood. The authors are confronted with the problem of making a double adjustment; first, to the child, and second, to those social forces and conditions that largely determine the nature and organization of the school.

New materials come into the curriculum in response to new needs, or in response to old needs spreading over wider areas, or whenever there is a redistribution of the burdens resting upon established institutions. Tradition and reason have not set a fixed division of work upon the various social institutions. As society becomes increasingly conscious of the value of the incidental agencies of informal education of one century, it frequently imposes them upon the schools. For this reason the new arithmetics contain numerous problems concerning the saving and lending of money, banking, building and loan associations, bonds, taxes, public expenditures, and insurance. Moreover, there is an increasing tendency for such problems to be arranged and presented in the grammar grades to correspond to the occupations they represent. Thus, we have a list of problems dealing with agriculture, another with banking, another with household science, and another with trade conditions. This is one evidence of the numerous attempts to adjust the school to vocational life.

The material thus arranged gives the student valuable

and interesting information, and it does not diminish his opportunities for acquiring arithmetical skill. Arithmetic is not taught primarily to give information about the Keokuk Dam or the Panama Canal, but in understanding and comprehending the Keokuk Dam and the Panama Canal some arithmetic is necessary. The solution of occupational problems does not rationalize arithmetic; such problems merely show a correlation of material in one field of thought with outside events, forces, and conditions. Involved in this attempt to emphasize the social and industrial phases of arithmetic are two highly desirable outcomes, (1) a knowledge of a more extended application and use of arithmetical forms and skills to the work of the world, and (2), a better appreciation of the values and uses of these forms and skills in aiding us to interpret our daily experiences.

Writers of arithmetic now make but little effort to localize materials. This is clearly the province of the teacher. It is easily possible for teachers to carry the practice to an unwise extreme. The authors knew of a teacher who lived in a community that gave considerable attention to poultry raising. Under the direction of this teacher the pupils devoted one entire term to the preparation of original problems dealing with poultry raising. Such extravagant attempts at local adjustment are regarded as ridiculous by most teachers and superintendents.

Abstract and Concrete Problems. Problems are commonly classified as abstract or concrete. There is a tendency to increase the number of concrete problems in the textbooks. In the lower grades these problems are selected from play, school, and home activities. The attempt is made to introduce no problems that are more difficult or complicated than those the children use in their daily life out of school. In this there is serious danger — that of keeping children upon a low level of arithmetical ability

until outside necessity forces them to acquire new skills. The problems used in school should frequently be somewhat more difficult than those met in life out of school; they should require some "stretching up." Otherwise the school is not fulfilling its purpose as a time and labor-saving device.

On the other hand, some have an erroneous notion as to what constitutes a concrete problem. Whenever simple abstract numbers are labeled with some denomination, like dollars, pounds, or acres, it is assumed that the problem has been made concrete. Such an assumption may be far from the fact. No matter what the labels may be, unless the facts of the problem have some basis in the child's experiences, the problem is more or less abstract to him.

There is still a third way in which the tendency to make arithmetic more concrete may lead to harmful, if not wasteful, results. It is the practice of labeling all the numbers in the process of calculation. The following examples will make this clear.

1. Chester had 13 marbles and his father bought 15 more for him; how many had he then?

2. A man bought 7 head of cattle at $45 a head; what was the total cost?

3. If $56 are divided among a number of boys so that each boy receives $7, among how many boys is the money divided?

4. If 30 oranges are divided equally among 5 boys, how many will each get?

(*a*)	(*b*)	(*c*)	(*d*)
13 marbles	$45	$7)$56	5 times)30 oranges
15 marbles	7 times	8 times	6 oranges
28 marbles	$315		

Referring to the above solutions, Dr. Faught says: "The impression seems to prevail that the processes of calculation

are in this way made concrete to the child. These problems can be solved concretely; but this would require us to count the marbles or divide the oranges among the boys. The processes of calculation are intended to avoid this method of solution; they have their justification in the fact that they enable us to obtain certain numerical results more economically and quickly than by dealing with the actual things themselves. But is there no place for the concrete in the solution of problems? To be sure, the child must be led to see the process or processes implied in the problem by appealing to the concrete, to objects, drawings, etc., and even to the actual things themselves, if necessary. That is the vital point in the solution of a problem, and the one at which so many teachers fail. But after the process or processes have been determined, it is a question of the abstract — of symbols."

"The word *marbles* does not help the child in thinking that 5 and 3 are 8, or the $ in thinking that eight 7's are 56. Neither does he, nor should he, have a mental image of marbles or dollars in thinking these relations. To insist on the boy counting his marbles after each number in the first solution is just as foolish as to insist on the boy counting his marbles thus: One marble, two marbles, three marbles, four marbles, etc. Boys do not count their marbles that way."

Children and adults use the processes of calculation in the shortest possible way for finding numerical results. They do not stop to count the things; they seldom know or try to remember such principles as, "If the dividend and divisor are both concrete the quotient is abstract." Labels usually do not assist in calculations, and it is doubtful if they aid materially in making the process concrete.

Teaching How to Solve Problems. A teacher should be careful to see to it that the work with concrete problems is meaningful and that the problems themselves are stated in

language and in situations that are similar to those which the child will encounter in life out of the school. The teacher must carefully direct the attempts of the pupils to solve problems and must make certain that there is as little waste as possible in time and effort in learning the processes involved. Thorndike gives the following seven "golden rules" which should guide the efforts of teachers in teaching concrete problems:

1. Consider the situation the pupil faces.
2. Consider the response you wish to connect with it.
3. Form the bond; do not expect it to come by a miracle.
4. Other things being equal, form no bond that will have to be broken.
5. Other things being equal, do not form two or three bonds when one will serve.
6. Other things being equal, form bonds in the way that they are required later to act.
7. Favor, therefore, the situations which life itself will offer, and the responses which life itself will demand.

Problems Arising Out of Real Situations. In life out of school isolated problems rarely exist as such. Rather are they part of a situation involving a number of activities. Much of the problem work in some of our textbooks is unreal, since it is isolated from real situations and may have no carry-over to them when the child meets the actual situation itself. A problem of finding the cost of an automobile is stated in a textbook in some such way as follows: How long must a person save at the rate of $40 per month to pay for an automobile worth $795?

If a man were to encounter a situation similar to that given in this problem his activities would be much more complicated than those of the pupil engaged in solving this textbook problem. Many problems involving mathematical thinking would arise out of the situation. The solution would require much more than mere addition or division. There would be numerous decisions and adjustments necessary with the average man about to purchase an automobile.

It is clear, therefore, that considerable attention should be given in our arithmetic teaching to developing in pupils the powe: to deal with real situations similar to those he will meet in life out of the school. This will involve the best type of thinking and the development of the power to make decisions and computations similar to those made by adults. Clearly the situations should be on the child level, leading gradually to more mature levels, reaching finally that of the adult.

Activities for the various grades which will develop arithmetic needs exist in the class work of pupils and many others can be developed. Below are suggestions of a few possible activities.

1. Planning a school paper.
2. Running a grocery store.
3. Planning a party or picnic.
4. The cost of a school paper.
5. How does a speedometer work?
6. How is a bank operated?
7. The school bank.
8. What factories are in our city and what do they make?
9. Building a toy house.

Groups of Related Problems. It is not possible that all school work in arithmetic be developed on a project basis. Pupils will necessarily be required to have much contact with life activities in vicarious ways through their reading, through imaginary situations, and through other similar types of activity. Most of the problems that are found in textbooks are isolated problems. The child works a number of these and there is no relation between the different problems he may be required to work other than that can be grouped around a certain mathematical process.

Some of the newer arithmetics contain series of related problems dealing with a single situation. A typical series is given below.

PLANNING A SLEIGH-RIDE PARTY

1. How many pupils are there in the class?
2. How much will the trip cost?
3. How long a trip can we take?
4. How many can go in one sleigh?
5. Shall we have refreshments?
6. What should be the amount for each pupil?
7. How much will that make for the entire class?
8. What will the cost be for each member of the class?

Many other problems can be thought of dealing with this situation. Their aim is to develop in the child the power to think quantitatively in a certain situation. This should be done not only in terms of fundamental processes but also in terms of relations between sets of conditions.

Types of Undesirable Problems. Teachers often use problems which it is difficult to justify. As a basis for classification of undesirable problems some of the types discussed by Thorndike in *The Psychology of Arithmetic* are used. Examples of each of the following types are given for illustration.

1. Problems involving misleading facts and procedures.
2. Problems involving trivialities and absurdities.
3. Problems involving useless methods.
4. Problems whose answers would in life out of school already be known.
5. Problems involving ambiguities and falsities.
6. Problems involving fantastic situations.

There are so many real problems that can be used in the class work that these undesirable types should be eliminated. Teachers and authors should so state the problems in arithmetic that the mental reactions and activities of the pupil will be similar to those used in life out of school. It is a questionable practice to give a child fantastic problems to work merely because such problems will give him practice in some process. Problems should be simple, direct, and real.

Type I. Problems Involving Misleading Facts and Unusual Procedures.

a. A box held 144 dozen oranges. How many oranges did the box hold?

b. Mary is 9 years old. Her mother is four times as old, her father is twice as old, and her brother is one-half as old. How old is each?

c. There are 210 windows in a building. Each room has 5 windows. How many rooms in the building?

There is some misleading fact or element in each of the above problems. The usual box of oranges does not hold 144 dozen. In problem (*b*), the facts indicate Mary's father to be 18 years old and Mary 9 years old. In problem (*c*) it would be a peculiar building that had no windows in the halls and stairways.

Type II. Problems Involving Trivialities and Absurdities.

a. Grace took 1,984 steps while walking to school and Susie took 873 steps less than Grace. How many steps did Susie take?

b. There are 1,400 ducks on a lake. 45 flew away, how many are left?

c. If 3 yards are needed in one child's dress, how

many dresses can be made for the children in an orphans' home from 273 yards?

d. The Smith family took a trip of 203 miles. They were gone a week. What was their average speed per day?

Problem (*a*), above, besides being an absurd situation, must have been answered before the problem was stated. It would be impossible to know that Susie took 873 steps less than Grace unless one knew the number Susie took. Problem (*b*) calls up an image of a pond covered with wild ducks. Could anyone count 1,400 ducks on a lake, even if he wanted to do so? The difficulties in the other problems are obvious.

Type III. Problems Involving Useless Methods and Operations.

a. I have \$15,345.00 with which to purchase land. If land costs \$93.00 per acre, how many acres can I buy?

b. A newsboy earns $\$\frac{2}{5}$ on one day, $\$\frac{3}{10}$ on another day, and $\$\frac{1}{2}$ on a third day. How much did he earn in all?

c. When eggs sell for \$.55 a dozen, how many dozen can be bought for \$66.00?

d. In how many weeks can you save \$250.00 if you save \$.20 a week?

e. How many feet have 266 cows?

Referring to problem (*a*), it would be most unusual for one to decide to spend \$15,345 for land and then to buy an exact number of acres. This problem would involve a real situation if stated, "A quarter-section of land sold for \$14,880. How much was that per acre?" Problem (*b*) involves a use of fractions which never occur in this way in

life out of school. Similar difficulties can be noted for each of the other problems.

Type IV. Problems Whose Answers Would, in Real Life, Already Be Known.

a. Bertha bought a pair of shoes for $6.00. This was just $\frac{1}{4}$ of what she had in her purse. How much had she before she paid for the shoes?

b. Tom's uncle gave him $3.00 for Christmas. His father gave him 69 cents more than his uncle. How much did his father give him?

In problem (*b*) Tom must have known how much his father gave him before he could figure out that it was 69 cents more than his uncle gave him.

Type V. Problems Involving Ambiguities and Falsities.

a. If you know the price of $\frac{1}{2}$ pint of cream, how will you find the cost of one gallon?

b. A man owns a farm of 96 acres near a growing city. He wishes to divide it into eighth-of-an-acre lots. How many lots will he have?

c. A train travels 60 miles an hour. How far will it travel in 16 hours?

d. John lives 11 blocks from school. How far does he walk each day if he does not take his lunch?

The answer to problem (*a*) would not be found by any of the fundamental operations. Cream would be rather expensive per gallon at the same rate we pay for $\frac{1}{2}$ pint. Problem (*b*) does not take into account land for streets and alleys. In problem (*c*) few trains, if any, make a scheduled run at the rate of 60 miles per hour, and never for a period of 16 hours without stops of appreciable length. In problem

(*d*) if one really wanted to know how many blocks John walks each day, a good many other facts would need to be included in the statement of the problem.

Type VI. Problems Involving Fantastic Situations.

a. A dairyman had 84 cows. He placed an equal number in each of 7 pastures. How many did he place in each pasture?

b. Jack skated 7 miles up the river and Joe skated 6 miles down the river. How far apart were they?

c. If it cost $200 to take some animals 4,000 miles, how much will it cost to take them 28,009 miles?

d. Two men start from a certain point and walk in opposite directions. One walks at the rate of 5 miles per hour, the other at 7 miles per hour. How many hours before they will be 72 miles apart?

e. In a race the first boy takes $5\frac{4}{5}$ seconds for his part; the second boy $5\frac{3}{5}$ seconds; the third boy $4\frac{4}{5}$ seconds. How much less than $\frac{1}{2}$ minute was needed for the race?

In the first problem the answer would not only be known, but it would very rarely, if ever, occur that a dairyman would have 7 pastures and have just the right number of cattle to put an equal number in each — if he ever would want to do so. In the second problem it is not clear whether the answer desired refers to the distance the boys are apart at the end of their skating, or at some other time. Is the river straight so that the boys will actually be the distance apart indicated by the facts? Such an arithmetical operation would be better practiced with some real situation where there are no misleading or unknown factors. The other problems involve situations that are not only impossible, but if a pupil were to actually realize what he

was asked to do he might wonder why he was being given problems to solve which could never occur.

Oral and Written Problems. Racially, oral arithmetic must have preceded written arithmetic. Necessity invented the abacus to take the place of counters, pebbles, and fingers. After writing was invented and written problems were introduced a long struggle began, yet unsettled, as to the proper distribution between the two types of problems, oral and written. So long as the school remains sensitive to social changes this will remain an unsolved problem. No perfectly satisfactory adjustment can be made. Change of emphasis is inevitable. The number of oral problems has greatly increased in recent years, perhaps relatively more rapidly than the number of written problems. A person is considered seriously handicapped if he must resort to paper and pencil to solve most of the simple problems of his daily life. A moment's reflection shows how greatly oral problems have increased. Every store, the grocery, the bakery, the hardware, the clothing and general merchandise stores, sell numerous articles, formerly considered as luxuries, that are now commonly regarded as necessities. Consequently, the average person is compelled to solve a greater variety of oral problems than formerly. No doubt as society grows more complex and varied in nature the number of such problems will continue to increase. Writers of textbooks in recognition of this new demand now give lists of oral problems drawn from daily life.

Some teachers think that rapid accurate work on written problems implies a corresponding degree of speed and accuracy in oral problems. Investigations show that a person may be rapid in written work and slow in oral work. It follows as a matter of course that many oral problems are necessary, if facility and skill are expected in the purely mental operations.

We would be seriously at fault, if we failed to urge the necessity of rationalizing and explaining many of the mechanical operations and of avoiding inaccuracies of statement. Analysis by steps is needed to familiarize children with the reasoning processes involved in the solution of the various types of problems. The virtue of an analysis is not its length but its clearness and conciseness. Direct analysis means a greater reliance on mental work.

Problems Without Numbers. Problems without numbers help the pupil to develop clear imaging and accurate reasoning. The pupil must concentrate upon the reasoning processes, as he cannot plunge headlong into computation. Such problems tend to develop the ability to decide what operation should be used under given conditions. There is a growing demand for more problems of this kind. Any resourceful teacher can make an abundance of them. We give a few illustrative problems:

If you know the height of a flagstaff on which there is a mark, and how far it is from the ground to this mark, how can you find how far it is from the mark to the top of the flagstaff?

To find the area of a square, what must you know and what must you do? If you know the dimensions of a square, how can you find its perimeter?

If you know the area of a triangle and the length of the base, what else must you know and what must you do to find the altitude of the triangle?

If you know the perimeter and two sides of any triangle, how can you find the remaining side?

I know how many feet long and wide an excavation is to be, what else must I know and what must I do to find how many wagon loads of earth will be taken out in making the excavation?

I have some money; I know the cost of one large marble; how am I to find how many such marbles I can buy at the

same rate with my money and how much money, if any, I will have left?

B owns a triangular lot; he knows the length, in rods, of its base, altitude and hypotenuse. How can he find from these measurements, how many acres it contains?

John weighed a basket of wood; after it stood out in the rain he weighed it again. How can he find the weight of water absorbed?

I know the length of a line of fence, and the number of posts, counting the end ones. How shall I find the distance from one post to the next?

What must you know and what must you do to find how many times a wagon wheel will turn in going a mile?

If you know the length of the edge of a cube, what can you find and how would you do it?

A cow is tied to the corner post of a square lot that is fenced in. Given the length of the rope, how can you find the area of ground over which she can graze, if she is inside the lot? If she is outside the lot?

If you know a certain fractional part of a certain number, how can you find the number?

If you know how many quarts of milk a family uses each day and the price per quart, how can you find the amount of the milk bill for the family for the month of January?

What must you know and what must you do to find how many bushels of wheat a bin will hold?

If you know the depth of the rainfall on a given field for a given time, what else must you know and what must you do to find the weight of the water?

What must you know and what must you do to find out how many bunches of shingles would be used in putting a new roof on your house?

If you know how much a man's average expenses are for a month what else must you know and what must you do to find out how much he saves in one year?

If you know the cost of a ticket from your home to New York City, what else must you know and what must you do to find the average cost of the trip per mile?

What must you know and what must you do to find the cost of laying a cement sidewalk in front of your home?

What facts must be known and how can they be used in

order to compute the average attendance at your school for the year?

Such work as has been suggested above gives the pupils a better appreciation of what the various arithmetical processes do than they otherwise have. It emphasizes the importance of knowing exactly what is to be done before attempting to do it. It makes its first appeal to the understanding rather than to the memory.

Original Problems. Children should be permitted to "make up problems." This is another excellent device for stimulating vigorous thinking. The duplication of a difficult problem by a simpler original problem frequently clears away a difficulty. The best of the original problems might be written in a book called "Our Own Arithmetic." The best of the problems thus preserved may be used later for review purposes. Whenever the data involved in such problems are of an informational character they should correspond to actual conditions. For example, it is not wise to assume in a problem that the distance from New York to Chicago is 250 miles; that wheat sells for $6 a bushel and silk for $185 a yard.

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See *Suggested Readings,* pp. 34-38.

PART THREE

CHAPTER XI

PRIMARY ARITHMETIC

A mastery of number will be acquired by children by giving them experiences which require actual uses of numbers in real situations. Such situations arise in many of the activities of the classroom and on the playground. The first work with numbers in the primary grades must always be concrete in order that it may have meaning. Gradually the child may be led from the simpler processes to the more complex, from the concrete to the abstract. If this work is carefully directed a sound understanding and appreciation of the use of number will result. The control over the use of number will be real since it has always been applied in meaningful situations. The final outcome will be the ability to think accurately in quantitative terms.

Objectives of Arithmetic. Number systems were devised to enable us to control more easily that portion of our environment which has a quantitative aspect. The race has built up a fairly simple system of mathematical thinking by means of which it is possible to make in many situations exact statements of the relations among conditions and phases of life. This precision and accuracy has been one of the factors which has made possible the great progress in science and in the control of certain human

relations. On the other hand it is clear that much of our thinking would be more accurate if it were quantitative. To the extent that the school has not trained its pupils to think correctly and with facility in quantitative terms, to that extent the real function of mathematics has not been used. To develop this power the school must give the child training in thinking in quantitative terms in real life situations. This can be accomplished by building up in the children a control over number through a series of activities closely related to their own lives.

The young child has relatively little need of arithmetic in his own activities, yet these activities must be made the basis for the number work in the school. Through experience and training the adult has learned to some extent the uses of number in conducting the affairs of society.

The school is the specialized agency of society which has been assigned the task of transmitting in as economical a way as possible the accumulated experiences and wisdom of the race. The school must, therefore, organize in a systematic way the experiences the children are to have in order that they may rapidly acquire mastery of the meanings and uses of the racial inheritances.

Much of the teaching of arithmetic has had this purpose in mind. On the other hand, much of it has been directed to the building up of a logical series of rules and processes in accordance with the standards and outlines set up by expert mathematicians. Special emphasis has been placed upon the development of a high degree of skill and accuracy in the fundamental processes, without much attention to the use that is made of these processes in the practical affairs of everyday life. Complicated problems and processes have been taught, although it has been perfectly clear that they are never used by the vast majority of people. The science of arithmetic has received undue emphasis. Arithmetic should be taught in the elementary

school for the purpose of developing controls over methods by which many of the activities of life are carried on.

Within the past decade there has come from within the school, and from society itself, the demand that the emphasis in arithmetic teaching be placed on its applications to life, rather than upon arithmetic as a science. The *Report of the Committee on Economy of Time in Education* emphasized the fact that much of the content of the arithmetic course was out of date and was retained largely because of tradition. The result of this report, and of other similar investigations has been the rapid elimination of much useless and unnecessary material.

The emphasis in curriculum making recently has been placed upon the discovery of the arithmetic needs of various common occupations, with the purpose of making it possible for the elementary school to give the training in processes actually needed by people in carrying on their life work. Investigations are now being made to determine the best methods of teaching the subject.

Arithmetic is no longer being taught chiefly because of the mental training which it gives, but because of its importance in the economic and social affairs of men. The introduction of school banks is an example of how business interests are helping the educational authorities to give the children contacts in school with organizations that have been largely for adults in the past. The increased emphasis that is being placed upon problems in taxation, insurance, bonds, buying and selling, and budgets shows the trend of the times.

Bobbitt has pointed out that it is probably more desirable that people have sound opinions based on the comprehensive consideration in the schools of the economic, social, and political affairs of the day, than that they become expert in the manipulation of tables and abstract processes alone. One of the aims of all arithmetic teaching should be to see

to it tnat this is brought about by giving the children ample opportunity to discuss critically and thoroughly, as a part of the class work, not only the methods of solving problems related to life situations which are stated in textbook language, but to study these problems in the way they will be required to solve them in life out of school, that is, in their natural settings.

Nature of Primary Arithmetic. Primary arithmetic has been restricted by common consent to a knowledge and mastery of the fundamental operations as expressed in integers and in easy fractions. It is not vocational in the sense that its processes are named after particular occupations. This selection of subject matter is due largely to the preparatory character and value of these fundamental processes. They are the intellectual tools which all must use in their later life, no matter what occupation they choose.

Dominance of Methods in Teaching Arithmetic. Mathematical instruction was long dominated by the logical theories and forms of organization of men of science. Each fact was taught to show its relation to other facts of the subject, and not to connect it with some common activities of life. Courses of study were usually constructed in the office of the superintendent and handed over ready-made to the classroom teacher. Special and personal modifications so essential in reconstructing the experiences of children were seldom encouraged. For this reason teachers taught as they had been taught by men of science, or as they had been told to teach, and not as the exigencies of the situation demanded. Instruction was logical, not psychological. The measure of the value of a fact was its relation to the other facts in an organized scheme, and not its occurrence in daily life.

Logical Versus Psychological Method. To the educated adult the development of arithmetic appears to be a logical process. It appears obvious that one should begin by

teaching the simple combinations first and proceed to the more complex. There have been courses of study which required that all addition be taught first, then subtraction, then multiplication, then division. This is a purely logical form of organization, but it is not in accordance with the known psychological principles by which arithmetic knowledge is acquired by children. It has resulted in the teaching of abstractions before the child has been able to give them meaning. No attempt was made to adjust the work to the level of ability or maturity of the child. The reaction from this led to the development of the spiral system of teaching arithmetic. Here an effort was made to teach first the simpler elements of all processes, then at a higher level the elements of a more complex character, until, in the course of a few years, the subject was covered as completely as in the logical method. This attempt to construct courses of study more nearly on a basis of child development has been called the psychological method. There has been great progress in this during recent years. The primary emphasis has shifted from the organization of subject matter itself to the study of how children really gain mathematical knowledge, and how much of it they need in their activities. The result has been that there is less emphasis on formal drill and less material presented in the primary grades than was formerly the case. Teachers are being directed to give the child experiences in which arithmetic needs will arise and where he will come into contact with them in real situations. An interesting example of this point of view may be found in Bulletin No. 16, 1919, U. S. Bureau of Education, *The Kindergarten Curriculum.*

There are mathematical values in the building materials, and through the children's varied experiences in handling it, they are laying the foundations for the later discriminations of form, size, number and arrangement of parts. The teacher, however, will not stress these formal aspects

of the material, but whenever a child reaches out spontaneously for some mathematical value, she will satisfy the need. In addition to this interest in mathematics, which is not uncommon in young children, a child naturally gains some knowledge of mathematical values when form, size, number, and arrangement become conscious factors in carrying out a project which he has himself initiated.

Progressive courses of study now emphasize the importance of child activity in the lower grades. Attempts are being made to discover activities which require a maximum of mathematical work. Statements are found repeatedly in courses of study that the child must gain number experiences directly in order that he may develop meanings. He is to be taught to discover them himself. The teacher is to aid him in rationalizing his number concepts. When he has them fairly well established, and repeated needs for certain combinations arise, drill can be begun, and the child can be taught how to abstract from the concrete situation the real meaning of the symbols.

This study of child needs has resulted in a critical study of the content of the curriculum, and has reduced emphasis upon the supposed value of formal arithmetic drill as mental training. The doctrine of formal discipline had taught that intensive work on complicated arithmetic processes gave a fine type of mental discipline which carried over to the other fields of activity, not directly related to it. This belief resulted in vigorous opposition on the part of many teachers to the elimination of any of the content of the course, on the ground of its value in mental training, although it had little application to life itself.

It has been shown that there is little, if any, transfer to other types of mental activity, except where the elements or bonds concerned are identical. Drill on complicated problems is, therefore, waste. The drill should be placed upon the simpler processes needed in life. This appears

more necessary since the results of investigations show that pupils are woefully inaccurate in even the simplest combinations, and solve simple arithmetic problem situations with difficulty.

Another factor that has resulted in the discrediting of the theory of formal discipline has been the challenge of society itself. Business has demanded increased accuracy. It has demanded that increased emphasis be placed upon the types of activities needed in business itself. It has compelled the elimination of many of the unnecessary processes which were needed in the past but which are not needed now. Studies of vocational and other life needs are being made by educators to determine what should be taught, and it is becoming increasingly evident that the amount of pure mathematics needed by the average citizen is astonishingly small. The line of development in the immediate future will be the stress upon quantitative thinking in real situations, and decreased emphasis upon drill in the more complex processes, but greater skill in the basic combinations and methods.

Time to Introduce Arithmetic. Opinion differs as to the proper time for beginning the study of arithmetic. In some schools a definite part of the school day is set apart during the first school year for the study of numbers. In other schools such work is postponed until the second school year, and in a few schools no formal study of number is taken up until the third school year.

In the more advanced European countries, arithmetic is studied during the first school year and the tendency seems to be in that direction in this country. Until about one hundred years ago arithmetic was not taught to pupils just entering school. Many think that any regular and systematic attention to number is premature in the first school year, and that the time should be spent in widening the pupil's activities and knowledge through reading, ele-

mentary nature study, garden making, games, drawing, and constructive exercises. They contend that it is legitimate to introduce only so much of number work as will serve a definite end in these subjects. The work in arithmetic, they maintain, should be incidental and not formal and systematic in the first school year. Whenever a quantitative relationship presents itself in any school activity, they would attempt to make it clear. They maintain that such a plan tends to develop number ideas naturally, and does not force them, and hence makes the subject more attractive to the child.

Pupils are asked to note the number in the class, the number absent or tardy; to tell the number of pencils, or books needed for the class or for a given row. Games offer numerous opportunities for introducing number ideas, because of the necessity of keeping score.

Others contend that when a pupil comes to school he is just as anxious to learn the fundamental relation of numbers as to learn to read, and that it is as unwise to postpone one as the other. Too often incidental teaching becomes accidental and perfunctory teaching, and a pupil finishes his first school year with his number ideas but little advanced. It is wise to motive the number work in the first and in all other grades, but many teachers wrongly assume that interest in a subject for its own sake is a motive that should not be considered. Most young children naturally like arithmetic, but many of them soon come to dislike it because of poor teaching in the lower grades. One of the great wastes in arithmetic lies just here. The interest of many pupils in the study of numbers is deadened and, as interest wanes accomplishment fails, and the pupil finds that he has developed a positive dislike for the subject. In general, the methods of teaching in the primary grades are more carefully thought out and systematized than in the intermediate grades, but a great responsibility rests upon

the teachers of the primary grades for giving a pupil as good a start as is possible for him in the subject.

Desirable Types of Activity. In the primary grades there is much opportunity to teach number in the different subjects of the course of study. The teacher must bear in mind that teaching number need not be confined to an arithmetic period, nor should it mean drill in the fundamental processes alone. The teacher must deal with the quantitative phases of other subjects. Any activity which is concerned with materials, makes it possible to develop concepts of size, form, shape, weight, and other quantitative characteristics. Pupils like to measure and to construct things. Excursions in nature study make it possible to study distance and direction. Pupils should be taught number in a variety of situations in order that they may have the necessary generalized notion of the meaning of a given fact. The more vital and interesting the experience, the more accurate and clear cut is the concept. In the first grade much number work should be done informally and incidentally. The danger here is that work with numbers will be overlooked by an inexperienced teacher, or be taken for granted. A sound basis of number work in the lower grades is vital to the accomplishment of satisfactory work in the higher grades.

In the work on projects in the primary classes there is abundant possibility of problems arising which require number for solution. These may be made the basis for later drill. A partial list of activities suitable for these grades is as follows:

1. Helping the teacher to keep the classroom neat and orderly.
2. Building a little city, as part of the handwork.
3. Playing store.
4. The school bank.

5. Making an individual book of animals, birds, or flowers.
6. Measuring liquids and solids.
7. Planting a garden.
8. Selling old newspapers.
9. Organizing a party.
10. Making a kite.
11. Making a map of the school grounds.
12. Learning to buy the groceries.
13. Distributing the school milk supply.
14. Drawing a picture according to directions.

In all of these activities there is abundant opportunity for number work. For example, if a teacher were to use the activity of building a little city, the following group of related problems would be some of those that would arise.

1. What kinds of buildings do we find in our city?
2. How many houses shall we make?
3. How many stores?
4. How many streets?
5. How long should they be?
6. Of what materials should we make them?
7. Where can we buy the materials?
8. What can you bring from home to help make our city better?

There is practically no limit to the problems that can arise out of such a situation. Teachers should not attempt in this work to secure mastery of the number facts. The basic consideration here should be the development of number meanings. These can be made the basis for concentrated drill in a special period set aside for the purpose. The drill will then have value, and the outcomes will be worthwhile. Much of the drill period is now wasted, because

the drill is not purposeful. The child often studies combinations because he is told to do so, not because he feels any particular need for them in carrying on his own activities.

The *Twentieth Yearbook of the National Society for the Study of Education* lists many activities pertaining to various parts of the country. They make clear what many teachers have done to vitalize number work.

The Use of Objects. Successful primary work today is almost intuitively associated with object teaching. No good primary teacher would think of attempting to teach numbers without the use of objects. When judiciously selected these serve as good substitutes for real outside situations. At any rate they are the school's one best means of making concrete the abstract numerical concepts that, up to the middle of the nineteenth century, were taught by purely memoriter methods. Pestalozzi has been called the father of object teaching. This is correct only in the sense that he restored the use of objects to the schools. As children gain in power of independent thought, that is, in ability to organize and condense their experiences, the number of objects used in teaching decreases; but the use of objects is never entirely discontinued, for they are always of value in interpreting new situations.

Dr. Suzzallo's discussion of the materials of objective teaching is particularly illuminating. His first indictment is the artificiality of the materials employed. "Primary children count, add, etc., with things they will never be concerned with in life. Lentils, sticks, blocks and the like are the stock objects of the schools, and to a considerable degree this will always be the case. Cheap and convenient material suitable for individual manipulation is not plentiful. But instances where better and more normal materials have been used are frequent enough in the best schools, to warrant the belief that more could be done in this direction in the average classroom. The 'playing at store,' and

the use of actual applications of the tables of weights and measures are cases that might be cited."

His second indictment is that the materials have too restricted a range. The range of materials is almost directly dependent upon the resourcefulness of the teaching staff. When teachers and supervisors know what they want and need in teaching, there will be comparatively little difficulty in inducing boards of education to buy it.

His third indictment is the limited use of the materials on hand. "It is too frequently the case that the teacher will treat the fundamental addition combinations with one set of objects; e. g., lentils. In all the child's objective experience within that field there are two persistent associations — 'lentils' and the relations of addition. A wide variety in the objective material used would make teaching more effective, particularly with young children."

His fourth indictment is that the materials of objective teaching have been too narrowly interpreted. He advocates a more extensive use of pictures, both in textbooks and in teaching, and of geometric figures and diagrams, which are of value, though in a more restricted way, in extending the range of concrete experiences.

His final indictment is that unsympathetic and conservative teachers use objects in a highly artificial and disorganized way. On the other hand the resourceful and progressive teacher secures some unity even in the materials she handles; she uses a game, some community activity, some childish interest, permits the children to play store, or be conductors on street cars, etc.

The wasteful effects of undue emphasis upon unrelated objects may be shown by a description of an actual recitation. The number fact to be taught was 6+3. The children were led to see that 6 blocks + 3 blocks = 9 blocks. This was arrived at by putting 6 blocks in one group and 3 blocks in another, combining these groups and then

counting, after which the pupils in turn repeated the "story." The process was then repeated with marbles and the children told the "story," 6 marbles+3 marbles =9 marbles. Then followed beans, toothpicks, pennies, shoe-pegs, peas, apples, crayons, pencils, rulers. Then pictures of cherries, oranges, nails, oblongs, squares, and chairs were used. Following this they passed to remembered objects and repeated the appropriate "story" for adding birds, trees, boys, lemons, horses, potatoes, and bricks.

In this particular school the primary lessons were taught in this way throughout the entire year. It took a year — a whole year — to teach the number facts and relations from 1 to 10. Had these children been ignorant of these particular facts a few examples would have supplied them with an adequate basis for making the proper generalizations.

The Use of Games. Nature has developed the play of children as one of the means of aiding them to acquire control over their reactions and responses to the environment. It is important that children learn through play to co-operate with the other members of the social group. In young children desirable social characteristics are developed by letting them live in a socializing environment. Teachers of arithmetic have long made use of the idea of playing in aiding the children to fix number facts. Play is spontaneous. Children readily acquire numbers that are necessary to the playing of a game. Many of them acquire a mastery over counting, combinations, and other arithmetic fundamentals, and can begin to see a real use of numbers. Since the interest of the children is highly stimulated by play, increased effort may be put forth without noticeable effect of fatigue. Children are really drilling themselves, but it is a pleasant performance. Teachers should study the playground activities aad discover the games children play which involve number, and should aid them in overcoming any difficulties they may encounter.

Games which the children can play by themselves may be taught as a part of the class work. Special attention should be given to the securing of real interest by varying the games and making them snappy.

Number games should not be complicated or difficult, but simple and easy to learn. At first the playing of the game should be the immediate objective of the pupils and the teacher should use it as a means of developing number. As new games are developed, they may become more complex and become real number drills. Pupils will be stimulated to greater effort because of the desire to excel and to win. The group spirit of rivalry also is a powerful incentive. The teacher can gradually turn the directing of the games over to the children, after some training. From this experience they will develop valuable qualities of self-direction and leadership, which, under absolute teacher control, would not be possible.

Some Simple Games for Developing Number Meanings

Playing with blocks	Parchesi
Swinging	Dominoes
Choosing sides	Chess India
Pull away	Flinch
Bean-bag	Authors
Tenpins	Maypole dances
Marbles	Rhymes for counting
Hopscotch	Measuring water, sand, etc.
Odd and even	Rhythm exercises
Tag	Choosing a leader
Lotto	Jackstones

Simple Games to Use for Drill. When pupils have learned to play games involving numbers in which no special attention is given to drill and the fixation of combi-

nations, the teacher can gradually introduce games which begin to assume the form of drill, although the children are still engaged in activities on the play level. This can be done after a teacher has developed number concepts, in order to fix them in the mind of the pupil. As far as possible these should arise out of some general activity of the class. Games can well be used to supplement the regular drill that a child will be obliged to have in order that the combinations be well fixed.

It should be borne in mind that one of the important characteristics of good number games is that all children should have an opportunity to drill at the same time, and that, whenever possible, each child should drill on the combinations with which he has difficulty. In many number drill games the only child getting any benefit out of the work is the child reciting. The teacher can avoid this difficulty by grouping her pupils in such a way that the drill can be adjusted to their needs. The strong pupils can be given one type of work, the average another, and the weak still another.

Games for Number Work

Around the Circle. The digits are arranged around a circle like a clockface, upon the blackboard. Any number of these digits may be used. A number is then placed at the center, and the numbers in the circle are multiplied by it as rapidly as possible. If a pointer can be made to swing on a pivot, the game may be varied by taking the numbers on which the pointer rests. The game can also be varied so as to involve addition and subtraction.

Bird Catcher. A variation of this well-known game is to have children sit in a circle, each taking a number. The pupil at the center gives easy examples of numbers that form combinations. When the result is the number of any

pupil in the circle that pupil holds up his hand. When the result is a number already agreed upon, all hold up their hands.

Buzz. The members of the class count in turn. When a given number or any of its multiples is reached, they say "buzz" instead of number. Those pupils fall out who miss or who hesitate in their answers.

Climb the Ladder. Draw a ladder on the blackboard. Put a combination on each step, and let a pupil go up giving results as fast and far as he can.

Hide and Seek. Combinations are placed on the blackboard with the different parts missing as: $14+3=?$, $14+?=17$, $?+3=17$, $4\times?=24$. In getting the answer the pupils unconsciously repeat the combination and form the desired habit. This is a very desirable form of play, and it is characteristic of the kind of game that a teacher can make for herself. Instead of the symbol ? it may be advantageous to use n thus giving a valuable algebraic form that will be helpful at a later time.

Morra. This is an old Roman game and is played with enthusiasm by Italians. A group of children stand or sit in a circle. Each extends, at a given word, all or any of his fingers. An immediate estimate is made as to the total number. All are added then to see who is nearest right.

Roll the Hoop. Draw a wheel on the board, or use an actual wheel. Place number combinations for spokes, and, by stating results in order, play that you make the wheel turn as fast as you can.

The use of games opens the way for children to handle the objects. Instead of being passive listeners with the teacher manipulating the objects as she demonstrates a solution, they thus become active participants in the process.

NOTE: See "*Number Helps*" published by Rand, McNally and Company, for the best list of number games.

Drills and Games in Grades Two and Three

Up and Down. A long list of combinations is written on the blackboard. A child is called upon to give the results as quickly as possible. If he makes an error, his name is written at the place, and another child takes the floor and begins at the first. He goes as far as he can and his name is written where he makes an error. When a child goes entirely through the list his name is written in a list of winners. This creates much interest in a small class, because there will be time for a second and sometimes a third try by pupils who fail. Since this is the case, they will watch closely to get the correct answer to the combination they missed, when it is given by some one else.

Out and In. Arrange the pupils in a circle. Use combinations of proper size for flashing on cards. Going around the circle if one hesitates in answering, the next answers and the pupil who failed must step within the circle. He is allowed to resume his place in the circle any time he answers first when someone fails to answer his combination. The children within the circle are anxious to get out and the drill is equally divided.

Combination Contest. Each row in the room is a number-work team whose captain is the child in the front seat. Two opposing teams stand and form a semicircle, team one forming one-half of the semicircle and team two the other half. The teacher flashes cards and the pupils give the answers in turn, beginning at left of the semicircle and finishing at the right. When a child misses he gets the card containing the combination he missed. After a definite period of time, the score is taken. The cards held by members of the teams are counted. The number of failures for each team is recorded on the board. Pupils not holding cards take their seats. Others return the cards to the

teacher, saying the combination correctly as they do so. A short period of drill is then used to help the failing pupils master their difficulties.

This has been found quite effective as it does not eliminate the pupil who fails.

Playing Store. On a large cardboard paste pictures of school supplies, games, toys, or foods that the children like, with prices attached from 1 to 10 cents. For addition ask for the cost of two or more articles. For subtraction ask for the change from twenty cents or less.

Racing Game. One pupil is chosen from each row who competes in turn with each pupil in his row. The racer stands behind the pupil with whom he is competing. The teacher shows flash cards. The first to give the answer gets the card from the teacher. After each row has had its turn, the winning racers compete with each other.

Card Game. Give a card containing ten examples to each pupil in the front seat of each row. At the word "Go" these pupils go to the board and write the answers to the first example, and then give the cards to the children behind them. If the first answer is wrong, the next child must correct it before beginning his example.

Competition Game. For a rapid drill sometimes use a race between the boys and the girls. Tell them whether it is to be addition, subtraction, or multiplication, then choose a boy and a girl and write a combination on the board. If the boy answers it first, the scorekeeper marks the combination *B*, if the girl, he marks it *G*. When the race is over, count the *B*'s and *G*'s and keep the score from time to time.

Champion. Material: Flash cards or examples written on the blackboard.

Procedure: Divide the class into groups of four or five children. The pupils of the first group stand. The champion of the group is the child who can say the answers

first. Give other examples until only one child remains standing. Each group is tested in like manner. The champions of each group compete for the championship of the class.

Formal and Rational Methods. Emphasis in instruction has changed from the mass to the group, from the class to the individual, from the transmitting of vicarious experience to fact experience, from a single appeal to a variety of appeals. The educational center of gravity has shifted from the subject matter to the child. The teacher thinks less of her function as a transmitter of knowledge by repetitious exercises and more of the necessity of breathing personality into dead materials, of multiplying the points of contact between the materials and the minds of the children, of permitting children to learn through their own experience rather than through the dogmatic statement of book or teacher. The teacher tries to stimulate a feeling of need in the minds of the children. True leadership as expressed in educative methods, rather than coercive methods, characterizes the spirit of modern instruction. Drill, as shown in another chapter, is rational rather than formal.

Counting. Attention will now be directed to certain specific phases of arithmetic. The most fundamentally important thing for beginners is to learn to count. Counting consists in thinking objects serially related and, later, in thinking the symbols that represent those relationships. Consequently, it is not synonymous with measurement, for that consists of comparing or measuring objects alike in nature — as two lines, two sides of a room, some shorter, some longer. The most fundamental mathematical concept depends upon a recognition of the one to one correspondence between objects. The expression of such correspondence is counting. Primitive people use a variety of objects in counting — pebbles, sticks, stones, toes, hands,

fingers, shells, grains of corn, notches on sticks, knots in strings. Their idea of number was expressed through the correspondence of the thing with the number of notches (cut in a stick) or knots (tied in a string) and names to express the correspondence were limited or wanting. With the invention of names counting began.

The facts of number are fixed through counting. Children learn that 17 is less than 18 and more than 16. The operations of addition, subtraction, multiplication, and division are attempted refinements of complex forms of counting.

Number Symbols. The next problem, after children have acquired the number concepts through the use of objects, is that of teaching them to read and write number symbols. The problem that the teacher has at this point is precisely the one she has in teaching beginning reading. The pupils already recognize by ear the number names, and use them in speech, but they do not know the figures by sight and cannot write them. Sight recognition must precede the writing of the symbols. At the beginning there must be a definite association of the symbol and the written form with the idea through objects. This association has usually been made by the children themselves, so that formal instruction of this character may not be needed at all. Those courses of study that set the limits from 0 to 10 for the first term or year do not take advantage of the mental equipment of the children. The process of reading and writing figures is in itself interesting. The prevailing tendency, however, is to associate the number series with a game or some interesting activity. If any instruction in the place relations of numbers apart from objective associations is necessary, it should follow the objective presentation of the numbers. Instantaneous recognition and response should follow the display and preception of figures.

Notation. The Arabic notation is a method of writing and combining numbers, giving them value according to the place they occupy. It is a simple device, an artificial method of remarkable significance. The reading of numbers of large denominations presupposes a knowledge of notation. When pupils have learned the number language up to 10, this may be extended rapidly from 10 and 1 to 10 and 9, and from two 10's to nine 10's and 9, and so on. When they can count and write to 100, the teacher may introduce units, tens, and hundreds; when they can read and write to 1,000, she may introduce thousands, and so on. A detailed application is seen in addition. When children can add single columns to sums of 9 or less, and can write to 100, they may be taught the adding of double columns without carrying. In teaching them the reason for writing one figure under another, the teacher may proceed by saying, "Let us see how many 10's there are in 48." The class discovers that there are 4 tens and 8 units; this explains why 48 is written as it is. The teacher next says, "I want to write 41 under 48. Let us see how many 10's there are in it." The children count ten, twenty, thirty, forty — four tens and one unit. It is then easy to see that if we want to add all of the ones (units) together and all the tens together, the most convenient arrangement would be to write them in columns.

Roman Numerals. But little attention should be given the Roman numerals, for two reasons: (1) They are rapidly going out of use. (2) They are more difficult to learn and cumbersome to write than Arabic numerals. Some years ago Professor Robert M. Yerkes of Harvard conducted a number of experiments to determine the time required to write Roman numerals and the number of errors made in using them. He found that it takes three and one-third times as long to write the Roman numerals from 1 to 100 as the Arabic, and the chance of error is twenty-one times

as great. It takes three times as long to read the Roman numerals from 1 to 100 as the Arabic, and the chances of error are eight times as great. In view of their limited use, there is no excuse for teaching any Roman numerals except those from one to twenty, fifty, one hundred, five hundred, and one thousand.

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CHAPTER XII

TEACHING THE FUNDAMENTALS

In the primary grades the pupils learn many number facts. In many courses of study teachers are instructed not to place special emphasis on drill, other than that found in games and similar activities, until the third grade is reached. However, when several number ideas have been supplied through concrete experiences, the teacher can direct her energies to the careful development of combinations and processes built directly upon them. The child can now be taught how to abstract number from concrete situations. The problem of the teacher is to aid the pupil in acquiring the necessary facility in handling the fundamental operations with speed and accuracy. The pupil must also acquire the ability to apply these processes in number situations that arise in life. Courses of study, therefore, give special attention to the careful development of well motivated, systematic drills to fix number habits. It has been found necessary to continue this work throughout the grades.

One of the chief problems of the lower grades has been to secure a satisfactory degree of accuracy. Often this has been impossible, because the drill work has not been properly done. The result has been that pupils have weaknesses which are hard to overcome, which, in fact, hinder them in all of their later work in mathematics.

The following operations are included among the fundamentals of arithmetic, because they are the ways in which number most frequently is used: Counting, addition, subtraction, multiplication, division.

The ability to add whole numbers is the chief number

need in the everyday social and economic affairs. As is shown in the chapter on *Waste in Arithmetic*, many textbooks provide for intensive drill on types of work in the fundamentals that are seldom used in business or industry. Children will be found who have not mastered even the simplest combinations.

The Serial Relation of Numbers. The school affords numerous opportunities for teaching counting; for example, the number of children in the room, the group, or class; the number absent; the chairs in the room; pencils in the box; the selection of certain objects from other objects; the counting of objects both seen and touched; the counting of sounds (taps); the repetition of an act. This may be followed by more abstract work when children count forward and backward, and later count by tens, fives, twos, and fours. When these have been begun in a more formal way, the number scale may be extended, and other numbers used in a similar way. The series idea, which is the basis of the number concept, may be abstracted very early in the school experience of the average child.

It is unnecessary to teach every number objectively. A pupil soon comprehends that 16 or 23 or any number occupies a certain place in a series. He does not think of it objectively. For this reason the number scale should be taught and memorized as such.

The structure of the number scale may be shown in the grouping involved in counting by tens.

0	10	20	30	40	50	60	70	80	90	100
1	11	21	31	41	51	61	71	81	91	
2	12	22	32	42	52	62	72	82	92	
3	13	23	33	43	53	63	73	83	93	
4	14	24	34	44	54	64	74	84	94	
5	15	25	35	45	55	65	75	85	95	
6	16	26	36	46	56	66	76	86	96	
7	17	27	37	47	57	67	77	87	97	
8	18	28	38	48	58	68	78	88	98	
9	19	29	39	49	59	69	79	89	99	

Counting by ones and tens may be extended to counting or adding by twos, threes, fours, fives, etc. These exercises lie at the foundation of good work in addition. Oral drills of this kind should be given until all sums of this character can be given accurately and rapidly.

In the reading of whole numbers, do not allow pupils to use *and* as one hundred *and* eighteen. This should be read one hundred eighteen. The *and* will be needed later to indicate the decimal point.

Addition. While a child has been learning the serial relation of number, the counting by groups has been teaching him the basic concept of addition. The transition from counting to adding must be made with great care, otherwise the child will learn many faulty procedures, such as counting with his fingers, his feet, tapping or in other ways he may invent. The simple combinations should at first be taught through the use of objects. Then the combination may be written on the board or printed on a card, the objects themselves not being seen, and the pupil be required to give the results without counting.

The teacher should be especially careful to make certain that the child gives the correct answer each time, so that faulty number habits may not be started. There is some evidence that errors in number combinations persist in much the same way that the correct concepts do. A child who establishes an incorrect bond will have considerable difficulty in overcoming it. If a pupil hesitates in giving the answer to a combination, he should be told the answer at once. When the correct bond has been established, the next problem of the teacher is to see to it that the response is habituated through well-motivated drill.

A convenient way of expressing the one hundred combinations is as follows.

TABLE VI

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

The first vertical column at the left of the table and the row at the top of the table contain the units which make up the fundamental combinations. The other numbers in the table are the sums of these combinations. For example, if 0 in the left-hand column and 0 in the top row are added the sum is 0; if 0 and 1 are added, the sum is 1.

The addition combinations are not all of equal difficulty. Some require but little drill before they become fixed, and others require extended repetition before they become automatic. The amount of drill required also varies with the child. Some children will find one combination easy while others will find it difficult. The teacher must, therefore, test carefully to discover the particular difficulties of each pupil, and then provide for individual needs. Investigations indicate that the order of difficulty of the combinations is not the same in concrete problems as in abstract settings. As part of the drill exercises the teacher should, therefore, give the pupils opportunity to make use of the combinations in concrete problems. This will from the beginning more nearly insure the ability of the child to apply number combinations in real activities and will enliven the class work.

In these problems the teacher should emphasize the idea that addition is the process of combining units of the same

denomination. Such quantities as *gallons* and *dollars* cannot be added. Such quantities as *yards*, *feet*, and *inches* can be changed to like units and then added.

The following rules which may serve as a guide in all drill exercises in arithmetic are given by Monroe in the *Eighteenth Yearbook of the National Society for the Study of Education*, Part 2, page 79.

(*a*) The child should understand the meaning of the combinations which he is memorizing.

(*b*) Attentive repetitions are necessary to fix the associations in learning the tables. These repetitions may be given either in isolated drill or in the use of these number facts in the doing of examples.

(*c*) For permanent memorizing (which is desired in this case) the repetitions of drill must be carried beyond the point where immediate recall is just barely possible.

(*d*) The learning should be done under some pressure or concentration.

(*e*) Memorize the number facts in groups, not one fact at a time.

Difficulty of the Combinations. The teacher who is developing the one hundred elementary addition combinations should provide amounts of drill on them according to their difficulty. The combinations that are hardest should receive most repetition. The combinations and their reverse forms can be grouped as follows:[1]

(*a*) Very hard

8	7	6	6	6	5	5	5	5
8	8	9	8	7	9	8	7	6

(*b*) Difficult

9	8	7	6	3	3
9	8	7	6	9	8

[1] Davis — *Technique of Teaching*, p. 211.

(*c*) Average

9	9	9	8	8	7	7	7	6	6
7	4	2	4	2	4	3	2	4	3

(*d*) Easy

5	4	3	2	2	1	2	3	4	5	6	7	8	9
5	4	3	3	2	1	1	1	1	1	1	1	1	1

(*e*) Very easy (often not included in the combinations)

1	2	3	4	5	6	7	8	9	0
0	0	0	0	0	0	0	0	0	0

Many writers have not included the zero combinations in their basic list. They have also omitted all combinations of the units with 10. The reverse forms of combinations have not been stressed. Instead teachers have been trained to teach what has been called the "forty-five basic combinations."

It has been found that if a child knows a certain combination, it does not follow that he will know its reverse. Teachers should, therefore, teach both forms of each combination. For example, both 3+2 and 2+3 should be taught.

Special attention should be given to the zero combinations. The child should learn to recognize these combinations as readily as he recognizes the words in a reading exercise. He should also learn to add in the following form:

$$1+1=2 \qquad 1+2=3 \qquad 2+4=6$$

The combinations can be grouped according to their sums as follows to aid in learning them.

$$\left.\begin{matrix}1+3\\2+2\end{matrix}\right\}=4 \qquad \left.\begin{matrix}1+6\\2+5\\3+4\end{matrix}\right\}=7$$

The teacher must remember that each of the number combinations is a specific fact, and must be learned as a fact, just as a child must be taught to spell many individual words in order to be certain that he will be able to spell them correctly. Each combination should be tested and taught until it is established.

Bridging the Tens. When a child has learned the fundamental combinations with a fairly high degree of speed and accuracy, and can add three numbers whose sum does not exceed 20, in a column, he should be taught the next step needed to enable him to work examples in simple column addition whose sum is over 20.

The combinations may be grouped according to their sums as shown below, to the right. For instance, if a child knows the sum of the combination of 5 and 4, as shown in (*a*), the combination of 15 and 4, as shown in (*b*), and the other combinations shown as (*c*), (*d*), (*e*), and so on must be taught because they can be solved by using the same basic number fact. The child can be taught to generalize from this series. When the child understands the process, the necessary drill can be given to fix the habit. As shown below, the device indicated is a good one to use for fixing this habit. First, the child should be taught to add 1, 2, 3, and so on, to each number in the first column.

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
5	15	25	35	45
4	4	4	4	4

30		1
36		2
39		3
33		4
32	and	5
35		6
37		7
38		8
31		9
34		10

Then, when he has learned the method for one series, the combinations given below can be used.

39		1	54		6
27		2	73		7
46	and	3	71	and	8
25		4	88		9
62		5			

Introducing Column Addition. The introduction of examples involving the column addition of three or more numbers whose sum is over 20, can take place after the child has learned the method of adding by endings. Short columns should be used first until the method of adding numbers in single columns becomes clear to the children. Then two columns can be used where no carrying is required. Carrying in column addition should then be taught. It is desirable that this process be rationalized for the child. However, children cannot follow as sustained a reasoning process as adults, and an explanation that may appear clear to a teacher may not be understood at all by a child. A teacher should make certain that the child knows the method of the process, and can carry it on consistently and

accurately. Children should be taught to check their work by adding first up the column and then down. They should be taught not to skip around in the column to find easy combinations to add, or to find 10's, but should learn to add the combinations as they come in a column. This can be possible only when the child is thoroughly familiar with the fundamental combinations.

In column addition, pupils often have difficulty in adding the second combination. In the example here given a child might be able to add the 5 and 4 and then have difficulty with the resulting 9 and 7 combination. This is probably in large part due to the fact that the child does not see the 9, but must carry it in his mind. To overcome this difficulty the teacher can use some such exercise as the following: Let the children have in mind some number, such as 18. The teacher can then write on the board or give orally a number to be added to 18 and have the children either write down the sum, or give it aloud. A small amount of this type of drill will establish the process for the child.

6
7
4
5

Many children acquire peculiar methods of work in attempting to overcome difficulties which they encounter. A teacher must always be on the lookout for counting, tapping, roundabout procedures, breaking up of number combinations into easier elements, and faulty statements in the solution of problems. As soon as these are discovered a teacher must begin at once the remedial work required to overcome them. Thorndike[1] says, "Unless the material for practice is adequate, well balanced and sufficiently motivated, the teacher must keep close account of the learning of pupils. Otherwise, disastrous underlearning of many bonds is almost sure to occur and retard the pupil's development. Practice that is actually in excess

[1]Thorndike — *The Psychology of Arithmetic*, p. 137.

is not a grave defect, if it is enjoyed and improves the pupil's attitude toward arithmetic."

Subtraction. Subtraction grows directly out of addition and some good teachers teach the two processes at the same time. There are one hundred basic combinations in subtraction. The following table gives these combinations, and it can be used for rapid oral and written drill. The combinations are given in regular order but they can be rearranged in any order that a teacher may wish.

TABLE VII

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

The child can be required to give the differences between any number in the table and the number in the first column at the left. The remainder is the number at the top of the column.

In teaching subtraction combinations the teacher can rationalize this work in the same concrete way that addition was made meaningful. A procedure which has been found helpful in this connection is as follows: All children in a group select a given number of objects, such as marbles or pieces of crayon to play the "story" of a combination. One child stands before the group to play the $\frac{7}{3}$ combination. He has 7 marbles in one hand and 3 in the other. He says, "7 marbles and 3 marbles are 10 marbles. 3 marbles and 7 marbles are 10 marbles. 10 minus 3 are 7." The teacher writes these combinations on the board as

they are given by the pupil. They can then be stated by the pupil without the marbles being used.

A number of plans for teaching subtraction, such as the complementary method, the borrowing and repaying plan, simple borrowing, the left to right plan and the "Austrian" method are described by Smith in the *Teachers' College Record* for 1919.

A child who has been taught a particular method of subtraction should be allowed to continue to use it even though he should move to some other school where it is not the standard procedure, otherwise he will be handicapped in his work. It is especially important that a uniform method be taught in all of the schools of a city, and by all teachers in a particular school.

When the child has learned the basic combinations in subtraction, he should then be taught how to apply them to the decades above the teens. The procedure used in teaching the bridging of tens is applicable here also, and can be used with good results. When this habit has been fairly well established the child can be taught to subtract two-place numbers from two-place numbers, at first with examples not involving borrowing, and then with those involving this process. The order of teaching should be as follows, one new process being introduced at a time.

First:	46	224	645
	32	121	230
Then:	136	158	339
	92	85	164
Then:	152	146	273
	33	28	165
Then:	225	311	303
	39	158	266

Multiplication. In multiplication in arithmetic we have a number to be used as an addend a given number of times. Multiplication involves three numbers, the multiplicand (the number to be repeated); the multiplier (the number showing the number of repetitions); and the product, showing the result.

The multiplicand and the product must be of the same denomination. This is not violated by such statements as $\$3 \times 2 = \$6$, because this should be read \$3 multiplied by $2 = \$6$.

The following will be found a convenient arrangement in connection with the tables and in counting by twos, threes, fours, fives, and so on.

It will be noted that the figures of the second column (Figure 1) represent the counting by 2's. This is also true of the second row. Those in the third row (or column) ascend by a constant addition of 3; those in the fourth by 4, and so on. The diagonal from A to C contains the figures 1, 4, 9, 16, and so on.

A											B
1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144
D											C

Figure 1

1	2	3	4	5	6	7	8	9	10	11	12
2											
3											
4											
5											
6											
7											
8											
9											
10											
11											
12											

Figure 2

The following number game is based upon the above arrangement. Have several pupils at the board (or seats) each prepare a large square containing the small squares as indicated in Figure 2. The teacher says "8." The pupils write this either under the 4×2 or the 2×4. The teacher says "40." This may be placed under 5×8, 8×5, 4×10, or 10×4; the teacher continues to state numbers and the pupils put them in the proper squares. The pupil who first fills all of the squares in a row or a column or a diagonal wins the game. The speed with which the numbers are announced by the teacher should be adjusted to the class, but care should be taken to insure that the numbers are announced rapidly enough to keep the pupils working near the limit of their ability. Several variations of this game will suggest themselves to teachers.

When a child has learned the one hundred multiplication combinations, which contain the same units as those for addition, multiplication of two places by one-place numbers without carrying can be begun, and then later with carrying. Then multiplication of two-place numbers by two-place

numbers, with and without carrying, should be taught. The order should be as follows:

I.	First: (Without carrying.)	23	31	24	51
		3	4	2	5
II	Then: (With carrying.)	24	44	36	29
		3	4	5	7
III.	Then: (Without carrying.)	22	21	32	42
		22	32	14	13
IV.	Then: (With carrying.)	26	45	37	92
		63	55	48	67

In each step the teacher must make certain that the child understands the process before the next step is taken up. Sufficient drill should be given to establish the procedure definitely, so that the pupil will not fail in any of the subsequent processes.

A child may know the tables perfectly, but fail on examples like those of types II and IV. This is because the child must keep in mind a number which is to be carried. In such an example as is shown to the left he must say: 5×7 equals 35; then 5×3 equals 15, plus 3 equals 18. The child must both multiply and add in an example of this kind. The following device has been found useful in eliminating this difficulty.

37
5

3×	8	plus	0	The child may begin with 3×8 equals 24; 24 plus 0 equals 24; and so on until he has finished the column. The order of the numbers in the column should be frequently changed. In this drill a teacher can avoid useless work by limiting the numbers a child is to add to the product to 1 less than the multiplier, since numbers larger than this never are added in an actual situation. For example, in multiplying by 4 the largest number that is ever added to a product is 3.
	7		1	
	3		2	
	2		3	
	0		4	
	4		5	
	9		6	
	5		7	
	6		8	
	1		9	

Special drill must be given on the zero combinations. These have been found to be among the most difficult in multiplication, largely because teachers and authors have neglected to give the required attention to them.

Division. Division is the analytic process of finding one factor when the other is known. A distinction is frequently made between division by partition and division by comparison or measurement. In partition a unit is to be divided into equal parts, the number of which is known and the size of each is required. The following is a problem in partition. Seventy-two books of the same kind cost 288 dollars; how much was that apiece?

Numerical solution: $\frac{1}{72}$ of \$288 = \$4.

Oral analysis: Each book cost $\frac{1}{72}$ of \$288, or \$4.

In division by measurement we have a whole to be divided into equal parts, the size of which is known and their number required. A problem in division by measurement is: A mother divided 8 cakes equally among her children, giving each 2, how many children has she?

Numerical solution:

$$2\overline{)8}\quad 4 = \text{number of children}$$

Oral solution: She has as many children as 8 cakes are times 2 cakes, or 4.

The numerical expression for measurement or division may be developed from subtraction in the same way that multiplication was developed from addition. Give a problem involving the division of a number (say 18) into equal parts, the size of each part being (say 6). Show that this same thing may be expressed more briefly.

In division there are ninety fundamental combinations in which there is no remainder. These are basic in the preparation for the work in short division and in work with examples having remainders. The ninety division combinations are as follows:

TABLE VIII

	0	1	2	3	4	5	6	7	8	9	10
1		1	2	3	4	5	6	7	8	9	10
2		2	4	6	8	10	12	14	16	18	20
3		3	6	9	12	15	18	21	24	27	30
4		4	8	12	16	20	24	28	32	36	40
5		5	10	15	20	25	30	35	40	45	50
6		6	12	18	24	30	36	42	48	54	60
7		7	14	21	28	35	42	49	56	63	70
8		8	16	24	32	40	48	56	64	72	80
9		9	18	27	36	45	54	63	72	81	90

In this table the number in the first vertical column at the left is the divisor. The numbers in the row at the top are the quotients. The numbers in the table are the dividends. For example, the numbers in the column under 2 are the numbers in which the numbers in the left-hand column are contained twice.

Short Division. Before beginning the more formal work in division a child should be able to tell instantly how many times 2 is contained in every number up to 20, and should state the remainder, if any. The same principle holds for 3 and 30, 4 and 40, etc. The most elementary facts of short division are taught in the learning of the

tables. The problem grows a little more complicated when the dividend is composed of several digits. But when the pupil is ready for this type of division, he should be familiar with the system of notation in use. With this as a basis, the explanation of the division of 6,234 by 2 is relatively simple. Whatever constitutes current practice gives us the clue to the form to be used in teaching short division. In the problem $2\overline{)6234}$ there is only one difficulty, i. e., the division of 2 into 30 or into 3 tens and 4 units. This difficulty is more apparent than real, for this very division has already been taught in connection with the tables. The truth is there is no special difficulty in teaching the facts and processes of short division, if the tables have been properly learned. The difficulty, such as there is, is in the notation.

Some pupils have difficulty with examples in short division in which there is a remainder. It is, therefore, important that pupils be given exercises in which they are required to state the quotients and the remainders. Class work, either oral or written, can be given by using some such device as the following:

10		1		
24		2		
46		3		
88		4		
25	divided by	5	equals	
36		6		
64		7		
72		8		
93		9		
58				

The pupils can be required to give the results, either orally or in writing, obtained by dividing the numbers in

the first column by those in the second. The numbers in both columns can be changed in any way the teacher may wish.

The order of introducing the different types of procedure needed in short division is as follows:

1. Simple even division.
   $4\overline{)8}$ $2\overline{)6}$ $3\overline{)12}$
2. Division without remainder in first step.
   $2\overline{)48}$ $3\overline{)366}$ $5\overline{)155}$
3. Division with remainder in first step.
   $4\overline{)172}$ $2\overline{)116}$ $7\overline{)175}$

When these steps have been well established, and the process applied, long division should be introduced.

Long Division. Many teachers teach long division as a process, and make little effort to explain the reason for each step. There are exceptions in long division, such as the 0 in the quotient, or difficult trial division, which are hard to rationalize so that children will understand them. These must be avoided when the process is first taught.

When long division is first presented, the teacher must make clear the steps in each division. They should be written on the blackboard or placed in some conspicuous place so that they may be readily referred to in case of difficulty. The steps can be indicated as:

(1) Divide
(2) Multiply
(3) Subtract
(4) Compare
(5) Annex

The first examples to be worked should be ones in which the first quotient is easily found. After considerable drill

on simple examples more complicated ones can be introduced until the procedure is clear to the child in all types of examples. The order may be somewhat as follows:

(1) $11\overline{)121}$ and $11\overline{)1221}$

(2) $21\overline{)441}$ and $21\overline{)463}$

(3) $24\overline{)504}$ and $23\overline{)544}$

(4) $42\overline{)4326}$ and $42\overline{)8517}$

(5) $38\overline{)684}$ and $38\overline{)695}$

(6) $28\overline{)4760}$ and $28\overline{)4666}$

(7) $66\overline{)3168}$ and $65\overline{)3000}$

Work should be given with examples with and without remainders. It is best to begin with examples in which there is no remainder.

Concrete Problems. In the early work all concrete problems should involve only one of the four fundamental processes. The same kind of problem should be continued until the form is fixed. The solution of written problems may be aided by, (1) a pictorial or diagrammatic representation of the problem, (2) an oral interpretation or estimate of the conditions of the problem before the solution is attempted, (3) the simplifying of the situation by using smaller numbers.

Encourage the pupils to make original problems illustrating the applications of the fundamentals. Problems may be based upon the prices of common groceries such as beans, sugar, coffee, rice, potatoes, flour, apples, bread; upon articles of clothing such as calico, muslin, linen, silk, collars, shoes, gloves, slippers, buttons; upon playthings; upon farm products; upon household furnishings; and the like. Interest will be stimulated if the pupils are encouraged to keep their original problems classified in a permanent notebook.

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CHAPTER XIII

SHORT METHODS

After the fundamental operations have been mastered, and the general underlying principles of arithmetic are understood, certain short methods may be introduced with profit.

It is not intended that the teaching of short methods should be postponed until the arithmetic of the grades has been practically completed, but it is best that before a pupil is taught a short method he should know the longer method of securing the same result. The longer method is usually more easily explained, and is more likely to appeal to the understanding of the pupil. If arithmetic were taught merely as a tool to be used in order to obtain correct results in computation, short methods should be introduced earlier than is here advocated.

How to Teach Short Methods. Short methods, if properly presented, will awaken much interest and enthusiasm, and will save much time in future computations. The teacher should carefully emphasize the fact that there is no short process in grammar school arithmetic that cannot be explained. Some of them are easily proved by the use of elementary algebra. It is neither necessary nor desirable that many of these proofs should be presented to a class in the grades, but the confidence of the pupil will be increased, if he is assured that a given short method can be readily explained and that it is not due to the so-called mystery of numbers.

A short method becomes really valuable to a pupil after he has such a mastery of it that he recognizes it under

whatever conditions it may be presented. A short method that has not been thoroughly mastered and impressed upon the mind by frequent drill is of comparatively little value. When an occasion arises in which the short method may be used to advantage, the mind trained to use the longer method fails to utilize the short method. A few of these short processes thoroughly mastered are of more value than many of them not well enough known to be readily used. Any short method that is learned will be forgotten within a short time unless the pupil has such a thorough mastery of it that he uses it almost automatically and unless his work is of such a nature that frequent opportunity to use the short method is presented.

Choose a few of the short methods which seem especially valuable, master them thoroughly, and be continually on the lookout for an opportunity to use them. When a few of them have been so mastered that their application has become habitual, a few more should be taken up in the same way. The importance of short methods can be easily over-emphasized.

Short Methods in Addition. Some rapid computers have habituated themselves to group numbers to 10 when adding. For example, when adding the numbers in the first column of this example, such computers would first group the 6 and the 4, then the 7 and the 3. Many rapid computers do not group the numbers to make any particular number but they make the additions in the order in which the addends appear in the example.

$$\begin{array}{r} 497 \\ 859 \\ 524 \\ 253 \\ 346 \\ \hline \end{array}$$

When adding such numbers as 37 and 46, the pupil in the upper grades should be taught to think $37+40+6$. Similarly, $53+36=53+30+6$.

Short Methods in Multiplication. Expert mathematicians use numerous short methods for obtaining results. Some of these, such as tables of logarithms, tables of

squares and cubes, and of square roots and cube roots, save an enormous amount of time in computation, but not all of these can be used to advantage by a pupil in the upper grades of the elementary school.

The short methods in multiplication which follow have already been presented, but they are grouped here for convenient reference.

To multiply by 10, move the decimal point one place to the right, annexing a zero if necessary.

To multiply by 100 or 1000, move the decimal point to the right two or three places respectively, annexing zeros if necessary.

To multiply by 5, multiply by 10, and divide by 2.

To multiply by 25, multiply by 100, and divide by 4.

To multiply by 75, multiply by 100, and take $\frac{3}{4}$ of the product.

To multiply by 125, multiply by 1000, and divide by 8.

To multiply by $33\frac{1}{3}$, multiply by 100, and divide by 3.

To multiply by 9, multiply by 10 and subtract the multiplicand.

To multiply by 11, multiply by 10, and add the multiplicand.

To multiply by 50, multiply by 100, and divide by 2.

To multiply by $12\frac{1}{2}$, multiply by 100, and divide by 8.

To multiply by $16\frac{2}{3}$, multiply by 100, and divide by 6.

To multiply by $66\frac{2}{3}$, multiply by 100, and take $\frac{2}{3}$ of the product.

To multiply by $37\frac{1}{2}$, multiply by 100, and take $\frac{3}{8}$ of the product.

To multiply by $62\frac{1}{2}$, multiply by 100, and take $\frac{5}{8}$ of the product.

To multiply by $87\frac{1}{2}$, multiply by 100, and take $\frac{7}{8}$ of the product.

To square any number ending in 5, multiply the number

of tens by 1 more than itself and annex 25. For example, to square 35. The number of tens is 3. Multiply this by 1 more than itself and the product is 12. Annex 25 and the final result is 1,225. By the same rule the square of 45 is found to be 2025. The square of 75 is 5625. Stated thus, $105^2 = 11{,}025$.

The above is obviously applicable, with modifications, to examples like the following: $(7\frac{1}{2})^2$, $(9\frac{1}{2})^2$, $(4\frac{1}{2})^2$. But it is easier to work such examples by the following rule: Multiply the integer by 1 more than itself and add $\frac{1}{4}$. Thus, $(7\frac{1}{2})^2 = (8\times7)+\frac{1}{4} = 56\frac{1}{4}$. $(9\frac{1}{2})^2 = (10\times9)+\frac{1}{4} = 90\frac{1}{4}$. $(4\frac{1}{2})^2 = (5\times4)+\frac{1}{4} = 20\frac{1}{4}$. It is evident that the $\frac{1}{4}$ may be written as .25.

To multiply two numbers whose tens digits are the same and whose units digits add to make 10. For example, 47×43, or 84×86. Multiply the tens digit by 1 more than itself and annex the product of the units digits. Thus, 47×43 is 2021; 84×86 is 7224; 13×17 is 221. Whenever the product of the units digits is a one-digit number, a zero must be put in tens place. For example, if 81 is to be multiplied by 89, the product of the units digits (1 and 9) is a one-digit number, hence the final result is 7209. Similarly the product of 51 and 59 is 3009. The rule for squaring a number ending in 5 is evidently a special case of the rule just cited.

To multiply two numbers between 10 and 20. For example, 17×15, or 14×17, or 19×16.

To either of the numbers add the units digit of the other. Annex zero to this result and add to this result the product of the units digits. Example, 17×15. $17+5$ equal 22 (or $15+7$ equal 22). Annex a zero and the result is 220. To this add the product of the units digits and the result is $220+35$, or 255.

Example, 14×17. $14+7=21$. Annex a zero and the result is 210. Add 4×7 to this, and the result is 238.

This computation may always be made orally with ease and is a valuable short method.

When one factor is as much greater than some multiple of 10 as the other is less than that same multiple.

Example, 28×32. 32 is 2 more than 30 and 28 is 2 less than 30.

Example, 46×54. 54 is 4 more than 50, and 46 is 4 less than 50.

Square the multiple of 10 and from this result subtract the square of the difference between one of the given numbers and the multiple of 10.

Example, 28×32. 30 is the multiple of 10 between these numbers. $30^2=900$. The difference between one of the given numbers and 30 is 2. $2^2=4$. $900-4=896$, which is the required product.

Example, 46×54. $50^2=2500$. $4^2=16$. $2500-16=2484$.

Example, 39×41. $40^2=1600$. $1^2=1$. $1600-1=1599$.

Those who understand the elements of algebra will recognize this short method as $(a-b)\ (a+b)=a^2-b^2$, since we may express 28×32 as $(30-2)\ (30+2)=900-4$ or 896.

The product of any two numbers is equal to the square of the number midway between them, minus the square of half their difference.

This comes under the same algebraic law as the short method preceding.

Example, $17\times13=15^2-4=221$.

Find the cost of 19 books at 13 cents each. Result in cents $=16^2-9=247$.

By the complement of a number is meant that number which added to the given number makes the next higher power of 10. Thus the complement of 98 is 2; the complement of 87 is 13.

To multiply two numbers whose complements are not large and whose complements are computed from the same power of ten, proceed as follows:

Example. 98×95.

98 2 = complement of 98.
95 5 = complement of 95.

98 − 5 and then annex 10, (5×2), 93 10 = required product.

From either number (98 or 95) subtract the complement of the other number. To this result annex the product of the complements.

Example. 89×93.

89 11 = complement of 89.
93 7 = complement of 93.

89 − 7 and then annex 77, (7×11), 82 77 = required result.

Example. 989×988

989 11 = complement of 989.
988 12 = complement of 988.

Subtract 12 from 989 and annex 132, (11×12) 977132 = required result.

When the complement is computed from 100 and the product of the complements is not a two-digit number a zero must be put in tens place. When the complement is computed from 1000 and the product of the complements is not a three-digit number, a zero must be put in hundreds place, or in hundreds and in tens places.

Example, 97×99. The product of the complements (3 and 1) is not a number of two digits. The result is 9603.

Example, $998 \times 994 = 992{,}012$ (2×6 is only a two digit number).

Example, $999 \times 997 = 996{,}003$ (1×3 is only a one digit number).

By the supplement of a number is meant that which subtracted from the number makes the next lower power of ten. Thus the supplement of 104 is 4; the supplement of 111 is 11; the supplement of 1007 is 7.

To multiply two numbers whose supplements are not

large and whose supplements are computed from the same power of ten, proceed as indicated below:

Example, 112×103.

112 12 = supplement of 112.
103 3 = supplement of 103.

$112+3$, then annex 36, (3×12), 11536 = required product.

To either number, 112 or 103, add the supplement of the other. To this result annex the product of the supplements.

Example, 117×105.

117 17 = supplement of 117.
105 5 = supplement of 105.

$117+5$, then annex 85, (5×17), 12285 = required product.

When the product of the supplements does not give a two-digit number, a zero must be put in tens place if the supplement is computed from 100. If the supplement is computed from 1000 and the product of the supplements is not a three-digit number, a zero must be put in the hundreds, or the hundreds and tens places.

Example, 102×104. The product of the supplements (2 and 4) is a one digit number. The result is 10608.

Example, $1014 \times 1003 = 1017042$.

After adequate drill the short methods involving complements and supplements can be applied quickly without the complements or supplements being written down.

To multiply any number by 11, proceed as shown below.

Example, 11×4532.

Write the number twice, thus 4532 and add.

Not infrequently the pupil will discover that in such an example as the following, it is a waste of time to multiply by the 9, then by the 5, then by the 4, and then add

the partial products. Pupils may be taught to note cases, such as this, in which we first multiply 7483 by 9, then multiply this product by 5, since 45 is 5×9, then add these products as here shown.

7483
459
67347
336735
3434697

Apply this method using the following multipliers:

819 728 486 549 497 328 568

To multiply any two numbers ending in 5:

Find the product of the two numbers at the left of the 5's, to this product add $\frac{1}{2}$ the sum of these two numbers, to this result annex 25.

Example, 65×85.

$6\times8 = 48$

$\frac{1}{2}(6+8) = 7$

5525

If $\frac{1}{2}$ the sum of the two numbers at the left of the 5's ends in $\frac{1}{2}$, drop the $\frac{1}{2}$ and annex 75 instead of 25.

Example, 35×65.

$3\times6 = 18$

$\frac{1}{2}(3+6) = 4\frac{1}{2}$

2275

To multiply two mixed numbers when the fractions are each $\frac{1}{2}$:

To the product of the whole numbers add $\frac{1}{2}$ the sum of the whole numbers and to this add $\frac{1}{4}$.

Example, $4\frac{1}{2}\times7\frac{1}{2}$. $4\times7=28$. $\frac{1}{2}$ the sum of the whole numbers (4 and 7) is $5\frac{1}{2}$. Add this to 28 and to this sum add $\frac{1}{4}$. The final result is $33\frac{3}{4}$.

Similarly, $5\frac{1}{2}\times9\frac{1}{2}$. $5\times9=45$. $\frac{1}{2}$ the sum of 5 and 9 is 7. $45+7+\frac{1}{4}=52\frac{1}{4}$.

Short Methods in Division. The following short methods

in division are placed here for convenience of reference. They have previously been presented.

To divide by 10, move the decimal point one place to the left prefixing a zero if necessary.

To divide by 100 or 1000, move the decimal point to the left two or three places respectively, prefixing zeros if necessary.

To divide by 5, multiply by 2 and divide by 10.

To divide by 25, multiply by 4 and divide by 100.

To divide by 125, multiply by 8 and divide by 1000.

To divide by $12\frac{1}{2}$, multiply by 8 and divide by 100.

To divide by $33\frac{1}{3}$, multiply by 3 and divide by 100.

To divide by $66\frac{2}{3}$, multiply by 3 and divide by 200.

To divide by $16\frac{2}{3}$, multiply by 6 and divide by 100.

CHAPTER XIV

COMMON FRACTIONS

Not many decades ago one who had mastered the subject of arithmetic through common fractions was regarded by many as a mathematician of no mean ability. Some still regard fractions as one of the most difficult subdivisions of arithmetic. Many young teachers have approached the subject timidly, and have grown discouraged as they attempted to teach children, who were blindly floundering their way through fractions.

The difficulties of the subject have been magnified because teachers have frequently emphasized the symbolism rather than the real concept of fractions, and because the subject has been presented as if an entirely new system of notation were necessary. Small wonder that many teachers have become discouraged and many pupils perplexed.

Children grasp truths most readily through contact with things. They weigh, measure, manipulate, and evaluate objects. It is through the actual handling of objects that the interpretive powers of children receive their initial training. Instruction must begin with the knowledge which children have. The limited knowledge of fractions which children have when they enter school has been acquired from the handling of objects common to their play and to the home. They do not think of $\frac{1}{2}$ and $\frac{1}{3}$ as symbolically represented, but as referring to particular things.

Early Introduction of Fractions. It is now customary to introduce certain fractions such as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ when the children are receiving their early instruction in integers.

The construction work of the primary grades is admir-

ably adapted for this purpose. The folding and cutting exercises of the lower grades enable the pupil to get the proper concept of two, three, and four equal parts of a unit, and they offer an excellent opportunity for the teacher to tell the pupil the names of these parts of a unit — a half, a third, and a fourth.

In the work of the early school years, there are other situations which give excellent and natural opportunity for the introduction of the simple fractional concepts. In the separating of rows or classes into divisions, in forming lines of march, in the use of rulers, in the distribution of materials, fractions naturally appear and under such conditions the fractional concept is easily grasped by the child. The alert teacher will not overlook the opportunity to use these natural situations.

When the pupil is taught that $2 \times 3 = 6$, the meaning of $\frac{1}{2}$ of 6 and $\frac{1}{3}$ of 6 can be easily made clear. Although the pupil works with such fractional units as halves, thirds, and fourths in the first four grades, the more formal presentation of fractions is usually deferred until the fifth grade.

Introductory Work. It is not possible for a child to get the proper conception of a fraction from the symbols. Many teachers have made the mistake of placing too little emphasis on objective work in fractions, and too much emphasis on the symbolism. The introductory work in this subject must be objective. It is of the utmost importance that the child shall actually *see* a half, a third, and a fourth just as he sees a foot or an inch. It is not enough that he be taught the name and the symbol, he must see the thing itself.

It is at least as important that the child be convinced by examples that a half is equal to two fourths, as that he should see that a quart is equal to two pints, or that a yard is equal to three feet.

Many teachers make the mistake of attempting to teach

the symbolism for common fractions before the child has really got the idea of fractions. The pupil who is properly taught will soon be able to name the fractional parts by noting the similarity of the number of equal parts and the name of the fractional unit. Thus, one of the *three* equal parts is called a *third*; one of the *four* equal parts is called a *fourth*, etc.

Instruction in fractions may begin by having the pupils divide some common object, such as an apple, into two or three equal parts. The child should then be asked to state into how many equal parts the object has been divided. When this has been repeated several times, attention should be directed to the units involved. Impress the idea that the whole apple is the original unit. A half of the apple is a fractional unit but it is just as truly a unit as is the whole apple. Similarly, $\frac{1}{3}$ of the apple is a fractional unit. This fact is not always made clear to pupils. There are integral units and there are fractional units. The fractional unit $\frac{1}{2}$ may be separated into fourths, and the fourths may be separated into eighths, sixteenths, or into other parts. Any part of a unit may be divided into equal parts just as the original unit was divided into equal parts to get that part. This idea may be most effectively taught by means of objects. Rulers on which the inch is subdivided into halves, fourths, eighths, and sixteenths, furnish good opportunity to familiarize the pupil with some of the common fractional units.

Stages in Teaching Fractions. There are three clearly recognizable and well-defined stages in the teaching of fractions. Their order is (1) the meaning, (2) the notation, (3) the manipulation. The meaning must be taught by the use of objects. When teaching the notation of fractions it is wise to emphasize the function which each term of the fraction serves. For example, after the pupil knows the meaning of the fraction two-thirds through objective pres-

entation, the teacher may write 2 on the blackboard and may ask the pupil whether this indicated 2 thirds. The pupil will at once see that the 2 alone does not indicate 2 thirds any more than it indicates 2 feet or 2 men. If we wish to indicate 2 thirds some further symbol is necessary. The pupil is then told that 2 thirds is written as $\frac{2}{3}$. From such a presentation, the pupil sees that the 2 shows the "how many," but it does not indicate what the fractional unit is. The 3 of the fraction $\frac{2}{3}$ shows what the fractional units are of which 2 are taken. After this conception has been thoroughly presented, the terms numerator and denominator may be given, and the teacher should explain why these names are chosen to represent the terms of a fraction.

To insure that the pupil has the proper idea, several fractions such as the following should be written, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{7}{10}$, $\frac{5}{16}$. Ask the pupil how many fractional units are denoted by each of these fractions and what the fraction unit is in each case. The teacher will be interested to know that in certain lines of business in which halves and fourths are very commonly used they are written without the denominator, which is understood. Thus, $3\frac{1}{4}$ is written 3^1, $3\frac{1}{2}$ is written 3^2, and $3\frac{3}{4}$ is written 3^3.

Definition of a Fraction. A commonly accepted but faulty definition of a fraction is that it is one of the equal parts of a unit. That this definition is incomplete and unsatisfactory is seen when we apply it to $\frac{2}{3}$ or $\frac{3}{4}$. These are more than "one" of the equal parts of a unit. No definition is adequate that eliminates the majority of the cases that should come under it. *A fraction is one or more of the equal parts of a unit.* This definition covers all the cases; it includes the separation of unity into equal parts, or of groups of integers, for example, 10, which may be divided into still smaller but equal groups.

The Ideas Involved in the Concept, Fraction. There are three ideas involved in the concept, fraction: (1) A unit

divided into equal parts, (2) a comparison of one or more of these parts with the original unit, (3) the grouping of these parts of the unit.

After the fundamental processes with fractions have been studied, it is necessary to emphasize the fact that a fraction may be thought of as an indicated division, and the fact that a fraction is an expression of the ratio of one quantity to another. The development of the fundamental processes with fractions is based on the concept of the fraction as one or more of the equal parts of a unit. It is very essential that by working with objects the pupils get the concept fraction so that a fraction is just as real to them as a whole number. To many pupils 3 feet and 5 pounds are real, but $\frac{3}{4}$ and $\frac{5}{8}$ continue to be but little more than symbols. Until the pupil really understands that a fraction is just as much a reality as a whole number, he is not ready to proceed to the further study of the subject.

The pupil has not gained the proper concept of fraction, as the term is used in the grades, until he sees a fraction as, for example, $\frac{3}{4}$ of a thing, $\frac{3}{4}$ of a group, and $\frac{3}{4}$ as large. Each of these will be discussed later. In many schools, great emphasis is put upon the first of these ideas and but little emphasis is given to the second and the third. The pupil knows that if he has 4 marbles and loses $\frac{1}{2}$ of them, he will have but 2 left. In this case the group under consideration was the 4 marbles and the pupil is concerned with the relation of what he has left to what he first had. He has lost $\frac{1}{2}$ of his marbles. The number of marbles left is $\frac{1}{2}$ of the original number.

Addition and Subtraction of Fractions. Some teachers present addition and subtraction of fractions in such a way that the pupil fails to see that the subject presents no new difficulties. The pupil who understands the concept fraction, and who understands fractional notation should have no difficulty with addition or subtraction of fractions.

Arithmetic is regarded by many as a difficult scubjet because many teachers do not present the subject in such a way that its essential unity is shown. To many a pupil, arithmetic consists of page after page of definitions, rules, and principles which seem to have but little relation to one another.

Addition and subtraction of fractions are based on exactly the same principle as addition and subtraction of integers, that only like things can be added.

If the processes in fractions are so presented to the pupil that but one difficulty at a time is introduced, and if the teacher is very careful to relate the unknown to the known, much of the supposed difficulty of fractions will disappear.

The Addition of Like Fractions. The pupil who has the proper understanding of what a fraction is should have no more difficulty in adding $\frac{1}{5}$ and $\frac{2}{5}$ than he has when adding \$1 and \$2 or 1 foot and 2 feet. If the teacher has properly emphasized the fact that the numerator indicates the number of the fractional units and that the denominator merely names the fractional units, the pupil will experience no difficulty when he needs to add fractions having the same denominator.

Give the pupil sufficient practice in the addition of like fractions, so that he will unerringly add the numerators to get the numerator of the result, and will then write this result over the denominator, which merely indicates the fractional unit. George S. Counts found in a study given to eighth-grade pupils that 60 per cent of the errors in addition of fractions were due to adding the numerators for a new numerator and then adding the denominators for a new denominator. For example, a pupil would say, $\frac{4}{5}+\frac{2}{5}=\frac{6}{10}$. A pupil who has been taught to add fractions as explained above and who has had sufficient well-motivated drill to fix the habit will not make such a mistake. Counts also found that 27 per cent of the errors made

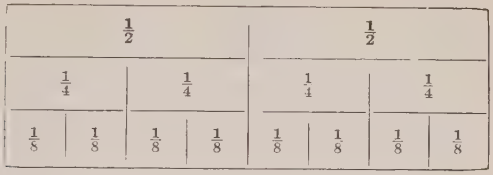
in the addition of fractions were due to multiplying the numerators for a new numerator and denominators for a new denominator. For example, the pupil would say, $\frac{3}{8}+\frac{5}{8}=\frac{15}{64}$. The pupil who does not thoroughly understand the meaning of fractions is very likely to say that $\frac{2}{5}+\frac{3}{5}=\frac{5}{10}$. To prevent this the teacher must teach more thoroughly the real meaning of the fractions and verify the addition by use of objects or diagrams.

Much of the actual use of fractions outside the school involves the addition and subtraction of fractional parts of an inch, a yard, a pound, a dozen, and similar common units. Many textbooks place excessive emphasis on the addition and subtraction of fractions of a kind which very few pupils will have occasion to use after they have left school.

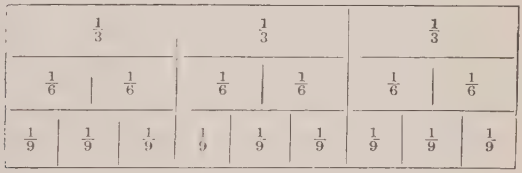
Adding Fractions of Unlike Denominators. Before the pupil finds it necessary to solve such a problem as $\frac{1}{2}+\frac{1}{4}$, he has learned that the sum of 1 inch and 1 foot is neither 2 inches nor 2 feet. He knows that before he can combine 1 inch and 1 foot, they must be expressed in the same denomination. Similarly, he knows that the sum of 5 quarts and 1 pint is neither 6 quarts nor 6 pints, and in the same way he may be led to see that the sum of $\frac{5}{8}$ and $\frac{1}{4}$ cannot be 6 eighths or 6 fourths. Before quarts and pints can be added it is necessary to express them in terms of the same denomination. Similarly, before fourths and eighths can be added they must be expressed in the same denomination.

At this stage in his work the pupil should study the relation of certain fractional units. The most effective way to do this is by means of diagrams and objects.

Such a diagram as the following may be used to show the relation of halves, fourths, and eighths.



The diagram below shows the relation of thirds, sixths, and ninths.



In the same way a diagram may be drawn to show the relation of such fractional units as halves, fourths, and twelfths.

The pupil should know the relationship between the fractional units commonly used, such as halves, fourths, fifths, sixths, eighths, and tenths, just as he knows the relation of pints and quarts, inches and feet, and the other commonly used tables of denominate numbers.

The addition of such fractions as $\frac{1}{2}$ and $\frac{1}{3}$ necessitates the changing of both fractions to fractions having a common denominator. The fact that such fractions must frequently be added offers motive for the introduction of this process.

The teacher should show by diagrams that if thirds are divided into two equal parts we have sixths; if thirds are divided into three equal parts we have ninths; if divided into four equal parts we have twelfths, etc. The pupil

should be led to see that thirds can be changed to any denomination which is a multiple of 3, such as 6, 9, 12, 15, 18. Similarly halves can be changed to any denomination which is a multiple of 2, such as 2, 4, 6, 8, 10, 12. To add $\frac{1}{2}$ and $\frac{1}{3}$ we must change both fractions and we seek a denominator with which both fractions can be expressed. The question then becomes, what is the least number that is a multiple of both 2 and 3. In the same way, if we wish to add $\frac{1}{3}$ and $\frac{1}{4}$ we must first find the least multiple of 3 and 4 that is common to them. The pupil should at this stage in the work be given adequate drill in finding by inspection the unit to which two or more fractional units may be changed.

By use of such diagrams as those on page 201, the pupil sees that when halves are changed to fourths, the fractional unit of the result is only half as large but twice as many of these fractional units are taken. When halves are changed to eighths, the fractional unit is only one fourth as large, but four times as many of the units are taken.

This leads to the statement of one of the fundamental principles of fractions:

Both terms of a fraction may be multiplied or divided by the same number without changing the value of the fraction.

For example: $\frac{3}{4} = \frac{9}{12} = \frac{3}{4}\frac{6}{8}$

and $\frac{6}{9} = \frac{4}{6} = \frac{2}{3}$

The principle of fractions stated above is also used in cancellation as explained later.

For use in the school it is better to state this fundamental property of fractions in two parts as shown below.

Both terms of a fraction may be multiplied by the same number without changing the value of the fraction.

Both terms of a fraction may be divided by the same number without changing the value of the fraction.

If the pupil finds it necessary to add $\frac{1}{2}$ and $\frac{1}{3}$ he knows

that these fractions must be expressed in terms of the same denominator and he knows that, for convenience, the denominator is the smallest multiple common to both 2 and 3. He finds by inspection that 6 is the lowest common denominator. The question now is, by what number must the denominator of $\frac{1}{2}$ be multiplied in order that the denominator become 6? In order that the value of the fraction shall not be changed both terms of the fraction must be multiplied by the same number, 3. Therefore, $\frac{1}{2}$ is equal to $\frac{3}{6}$. Similarly, to reduce $\frac{1}{3}$ to sixths, we find what number the denominator must be multiplied by to give 6, then we multiply both terms of the fractions by this number.

The pupil, having expressed $\frac{1}{2}$ and $\frac{1}{3}$ as sixths, is able to add them without further difficulty.

At this point drill should be given such as that suggested by the following.

If I wish to express $\frac{3}{4}$ as twelfths, by what number must both terms of the fraction be multiplied? If I wish to express $\frac{5}{6}$ as twelfths by what number must both terms be multiplied? What is the result in each case?

Express each of the following fractions as twelfths:

$$\frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{6} \quad \frac{5}{6}$$

Express each of the following as twentieths:

$$\frac{4}{5} \quad \frac{1}{2} \quad \frac{3}{4} \quad \frac{3}{5} \quad \frac{7}{10}$$

The pupil is now prepared to add such fractions as those most commonly used.

The addition of such fractions as $\frac{3}{4}$ and $\frac{1}{2}$ gives a result, $\frac{5}{4}$, which is an improper fraction. If the pupil will recall that the numerator shows the number of fractional parts taken and the denominator names the fractional part, he will be led to conclude that in $\frac{5}{4}$ there is more than 1 whole.

It takes 4 fourths to make a whole. Therefore, $\frac{5}{4}$ is equal to 1 whole and $\frac{1}{4}$ more. This is expressed as follows: $\frac{5}{4}=1\frac{1}{4}$.

After some exercises such as this the pupil is led to see that the answer obtained, when reducing an improper fraction, may be most quickly obtained by simply dividing the numerator by the denominator and this is the procedure that should be practiced until it becomes habit. For example, $\frac{25}{4}=6\frac{1}{4}$. $\frac{18}{5}=3\frac{3}{5}$.

Subtraction of Fractions. If the addition of fractions has been presented as previously explained, the subtraction of fractions will offer no difficulties. All of the principles needed to explain subtraction of fractions have been presented and the pupil is ready to proceed.

The first examples presented should be those involving the subtraction of like fractions such as $\frac{3}{4}-\frac{1}{4}$.

The next examples should involve such fractions as the following: $\frac{7}{10}-\frac{2}{5}$, and $\frac{7}{8}-\frac{3}{4}$.

The pupil is then prepared to subtract any of the commonly used fractions such as, $\frac{3}{5}-\frac{1}{4}$, $\frac{2}{3}-\frac{1}{5}$, $\frac{7}{8}-\frac{3}{5}$.

Counts found that two errors frequently occurred in subtracting fractions with like denominators. Many pupils subtracted the numerator for a new numerator and also subtracted the denominators for a new denominator. For example, a pupil may say $\frac{5}{8}-\frac{3}{8}=\frac{2}{0}$. Some pupils multiply numerators for a new numerator and multiply denominators for a new denominator. For example, a pupil may say $\frac{5}{8}-\frac{3}{8}=\frac{15}{64}$.

The subtraction of fractions frequently gives a result which may be expressed in lower terms by using the principle, "both numerator and denominator may be divided by the same number without changing the value of the fractions."

Fractions in Common Use. The fractions used in textbooks should be like those which are in common use out-

side the school. Many teachers are surprised to learn that almost all of the fractions used outside the school are very simple and that the denominators in most common use are 2, 4, 8, 16, 32, and 64.

Such a fraction as $\frac{3\,7}{4\,1}$ is almost never met except in textbooks and such a fraction as $\frac{4\,7\,7}{9\,2\,4}$ is a mathematical monstrosity. The arithmetics of some years ago contained many unusual fractions of this type, but the best of our modern texts rarely use fractions with denominators larger than 20 and most of the denominators used are not greater than 16.

Fractions with denominators of 7 or 9 are sometimes but not frequently used.

An intensive investigation made by Wilson indicates that more than 95 per cent of the fractions commonly used are included in the following list: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{5}$. The fraction $\frac{1}{2}$ is more frequently used than any other. Few problems outside the school involve more than one fraction. To meet the needs of the business and industrial world the treatment of fractions does not need to be extensive or involved.

Teachers will find it instructive to have the pupils investigate what fractions are in common use in the various commercial and industrial activities of the community. Such facts as the following will be discovered: Wire nails are made in the following lengths: The denomination in each case is inch. $\frac{5}{6}$; $\frac{7}{8}$; 1; $1\frac{1}{4}$; $1\frac{1}{2}$; $1\frac{3}{4}$; 2; $2\frac{1}{4}$; $2\frac{1}{2}$; $2\frac{3}{4}$; 3; $3\frac{1}{4}$; $3\frac{1}{2}$; 4; $4\frac{1}{2}$; 5; $5\frac{1}{2}$; 6.

The diameters of wires vary from $\frac{1}{2}$ inch to $\frac{1\,7}{2\,5\,6\,0}$ inches. Some of the larger sizes are as follows, expressed in terms of the inch. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{8}$, $\frac{7}{16}$, $\frac{5}{16}$, $\frac{3}{16}$, $\frac{1}{16}$, $\frac{1}{20}$, $\frac{1\,5}{3\,2}$, $\frac{1\,3}{3\,2}$, $\frac{1\,1}{3\,2}$, $\frac{9}{3\,2}$, $\frac{7}{3\,2}$, $\frac{5}{3\,2}$, $\frac{3}{3\,2}$, $\frac{1}{3\,2}$ and $\frac{1}{4\,0}$.

The pupils may be asked to find out by inquiry at a hardware store in what dimensions screws, bolts, bits, and shot are usually made. Inquiry at a dry goods store will

reveal the small number of fractional units used to express the widths of ribbon, and of dress goods. At the machine shop inquiry will reveal the fact that the denominations of almost all the fractions used are as stated above. The carpenter and the contractor, the architect, and the engineer make frequent use of fractions, but they use such fractions as those previously indicated.

Addition and Subtraction of Mixed Numbers. Two of the most commonly used forms in the addition of mixed numbers are shown below.

$$(a)\quad \begin{array}{r} 7\tfrac{1}{4}=7\tfrac{3}{12} \\ 9\tfrac{2}{3}=9\tfrac{8}{12} \\ 5\tfrac{1}{2}=5\tfrac{6}{12} \\ \hline 21\tfrac{17}{12}=22\tfrac{5}{12} \end{array} \qquad (b)\quad \begin{array}{rcr} 7\tfrac{1}{4} & = & \tfrac{12}{3} \\ 9\tfrac{2}{3} & = & 8 \\ 5\tfrac{1}{2} & = & 6 \\ \hline 22\tfrac{5}{12} & & 17 \end{array}$$

The first form shown above requires the rewriting of all of the whole numbers and the writing of each fraction expressed with the common denominator. In the second form the common denominator is written above the line and the numerators being written as indicated are easily added.

In many of our best schools the form suggested at the right is now used. By the use of this form it is not necessary to write the common denominator twice, as in form (*b*).

$$\begin{array}{r|l} 7\tfrac{1}{4} & \\ 9\tfrac{2}{3} & \\ 5\tfrac{1}{2} & \\ \hline & 12 \end{array}$$

The subtraction of mixed numbers offers no difficulty not already explained except when the fraction of the minuend is smaller than the one in the subtrahend and when a mixed number is to be subtracted from a whole number.

The form at the right is now used by many of our best teachers.

$$\begin{array}{r|l} 7\frac{1}{3} & \frac{1\,3}{4} \\ & \overline{16} \\ 2\frac{3}{4} & 9 \\ \hline 4 & \frac{7}{12} \end{array}$$

One-third is equal to $\frac{4}{12}$ and $\frac{3}{4}$ is equal to $\frac{9}{12}$.

As $\frac{4}{12}$ is smaller than $\frac{9}{12}$, add 1 or $\frac{12}{12}$ to $\frac{4}{12}$ which makes $\frac{16}{12}$. Nine-twelfths from $\frac{16}{12}=\frac{7}{12}$. Since $\frac{12}{12}$ or 1 was added to the minuend, 1 must also be added to the subtrahend in order that the remainder shall not be changed. It is more convenient to add 1 to the 2 which makes 3.

Some teachers prefer to use the following form in such cases:

$$\begin{array}{r} 7\frac{1}{3}=7\frac{4}{12}=6\frac{16}{12} \\ 2\frac{3}{4}=2\frac{9}{12}=2\frac{9}{12} \\ \hline 4\frac{7}{12} \end{array}$$

After sufficient practice in addition and subtraction of mixed numbers the best pupils in a class will find that they can do all of the work mentally.

Multiplication of Fractions

How to Multiply a Fraction by a Whole Number. The pupil has learned that when he multiplies by a whole number he is saving time. He may, for example, find out how much 8×4 ft. is by using 4 ft. as an addend 8 times, but the process of multiplication enables him to save much of the time that would otherwise be used in making the addition. Similarly 8×$4 is $32; 8×4 marbles is 32 marbles; 8×a unit of any denomination is equal to 32 units of that denomination. Therefore, 8 times $\frac{4}{5}$ is equal to $\frac{32}{5}$ and 8 times $\frac{4}{15}$ is equal to $\frac{32}{15}$.

The multiplication of a fraction by a whole number will

present no difficulties to the pupil who will recall the fact that the numerator indicates the number of units, and the denominator merely indicates the name or denomination of the units. If the teacher so desires, the pupil may be led to formulate the rule that *to multiply a fraction by an integer, multiply the numerator by that integer.*

The pupil who can multiply $8\times\frac{4}{5}$ will have no difficulty with $\frac{4}{5}\times 8$ as he has already learned that either number may be used as the multiplier when finding the product of two numbers.

Multiplication of Fractions. To find the product of $\frac{1}{2}\times\frac{1}{3}$ it is first necessary to show the pupil that $\frac{1}{2}\times\frac{1}{3}$ means $\frac{1}{2}$ of $\frac{1}{3}$. The method by which this may be done is explained in all good textbooks. By use of diagrams such as those shown on page 201 the pupil has learned that $\frac{1}{2}$ of $\frac{1}{3}$ is $\frac{1}{6}$ and that $\frac{1}{3}$ of $\frac{1}{8}$ is $\frac{1}{24}$. Hence, $\frac{1}{2}$ of $\frac{2}{3}$ is $2\times\frac{1}{6}$ or $\frac{2}{6}$, and $\frac{1}{3}$ of $\frac{3}{4}$ is $3\times\frac{1}{12}$ or $\frac{3}{12}$.

Suppose now that the pupil wishes to find $\frac{2}{3}\times\frac{4}{5}$. He knows that $\frac{1}{3}$ of $\frac{1}{5}=\frac{1}{15}$. Therefore, $\frac{1}{3}$ of $\frac{4}{5}=4\times\frac{1}{15}=\frac{4}{15}$. Therefore, $\frac{2}{3}$ of $\frac{4}{5}=2\times\frac{4}{15}=\frac{8}{15}$. It is, therefore, evident that to find the product of two fractions we multiply the numerators of the fractions for the numerator of the product and multiply the denominators for the denominator of the product. It is possible to show $\frac{2}{3}$ of $\frac{1}{5}$ objectively, but it is easier to develop the inductive procedure as shown above and then derive the rule for future guidance.

Attention should be directed to the fact that it is often possible to shorten the multiplication of fractions by use of cancellation. For example, $\frac{\cancel{4}^{2}}{5}\times\frac{7}{\cancel{8}_{2}}=\frac{7}{10}$. It is of great importance that the pupil recognize the fact that cancellation, as used here, is based on a principle already learned; namely, dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction. The pupil can easily be led to see that

$\frac{4}{5}$ of $\frac{7}{8}$ could be expressed as shown to the right, and the 4×7 be regarded as the numerator and 5×8 as the denominator. Many pupils have the erroneous impression that in cancellation one term of a fraction is divided by all or a part of the other term.

$$\frac{4\times7}{5\times8}$$

How to Multiply by a Mixed Number. If the multiplicand and the multiplier are both mixed numbers, and if the whole numbers are not large, time is usually saved by first reducing the mixed numbers to improper fractions. For example, $2\frac{1}{2}\times8\frac{1}{3}=\frac{5}{2}\times\frac{25}{3}=\frac{125}{6}=20\frac{5}{6}$.

When the whole numbers are large it is sometimes better to use the method shown at the right. The difficulty involved in this procedure is the correct placing of the partial product. The separate multiplications made necessary by this method frequently lead to confusion and the process is sometimes unduly long. Many pupils are somewhat troubled by the fact that in examples involving fractions the product is sometimes less than the multiplicand or the multiplier. While working with integers the pupil found that the product is larger than either of the factors and this fact leads to the difficulty.

$$\begin{array}{rl} 45\frac{2}{3} & \\ 47\frac{4}{5} & \\ \hline \frac{8}{15} & (\frac{2}{3}\times\frac{4}{5}) \\ 36 & (\frac{4}{5}\times45) \\ 31\frac{1}{3} & (47\times\frac{2}{3}) \\ 315 & \\ 1800 & \\ \hline 2182\frac{13}{15} & \end{array}$$

Teachers will find it wise to have the pupils derive and then state the following.

When a number is multiplied by a number larger than 1 the product is larger than the number.

When the number is multiplied by a number smaller than 1 the product is less than the number.

When a number is multiplied by a number equal to 1 the product is the same as the number.

Division of Fractions

How to Divide a Fraction by an Integer. The pupil will have no difficulty with such examples as $\frac{4}{5} \div 2$ and $\frac{6}{5} \div 2$. He knows that $\$4 \div 2$ is $\$2$ and $\$6 \div 2$ is $\$3$. He knows also that 4 of any unit divided by 2 is equal to 2 of that unit. He, therefore, concludes that $\frac{4}{5} \div 2$ is $\frac{2}{5}$ and that $\frac{6}{5} \div 2$ is $\frac{3}{5}$.

The pupil also knows that if he needs to find the quotient of $\frac{3}{5} \div 2$, this may be regarded as $\frac{1}{2}$ of $\frac{3}{5}$. By the use of diagrams he can show that $\frac{1}{2}$ of $\frac{3}{5}$ is $\frac{3}{10}$. In this case each of the fractional units has been divided into 2 equal parts.

The fact should be emphasized that to divide a number by 2 we take half of the number; to divide a number by 3 we take a third of the number, etc. For example, $\frac{5}{6} \div 2 = \frac{1}{2}$ of $\frac{5}{6}$; $\frac{7}{8} \div 3 = \frac{1}{3}$ of $\frac{7}{8}$.

How to Divide a Fraction by a Fraction. Just as $\$8 \div \$4 = 2$, and 8 ft. $\div$ 4 ft. $= 2$, so 8 fifteenths $\div$ 4 fifteenths $= \frac{8}{4}$ or 2, or $\frac{8}{15} \div \frac{4}{15} = 2$.

We do not divide feet by inches, but before attempting the division we express both dividend and divisor in terms of the same unit. From such illustrations as this the pupil may easily be led to the solution of such examples as the following:

$$\frac{3}{4} \div \frac{2}{3} = \frac{9}{12} \div \frac{8}{12} = \frac{9}{8} = 1\frac{1}{8}$$

$$\frac{2}{5} \div \frac{3}{4} = \frac{8}{20} \div \frac{15}{20} = \frac{8}{15}$$

The Inversion of the Divisor. There are a number of ways of explaining the rule for the inversion of the divisor. Some attempted explanations are undoubtedly beyond the ability of pupils of the fifth grade. The pupil may glibly repeat an involved explanation that he has memorized, but he may not understand it. Probably the simplest

method is to use the reduction of fractions to like units as shown above.

If this method is to be used the pupils should work at the blackboard quite a number of examples like the following:

$(a)\ \frac{2}{3} \div \frac{3}{4} = \frac{8}{12} \div \frac{9}{12} = \frac{8}{9}$ $(c)\ \frac{2}{5} \div \frac{5}{6} = \frac{12}{30} \div \frac{25}{30} = \frac{12}{25}$

$(b)\ \frac{2}{3} \div \frac{7}{8} = \frac{16}{24} \div \frac{21}{24} = \frac{16}{21}$ $(d)\ \frac{2}{5} \div \frac{3}{4} = \frac{8}{20} \div \frac{15}{20} = \frac{8}{15}$

Now ask the pupils to rewrite the above examples, omitting the intervening step, as follows:

$(a)\ \frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$ $(c)\ \frac{2}{5} \div \frac{5}{6} = \frac{12}{25}$

$(b)\ \frac{2}{3} \div \frac{7}{8} = \frac{16}{21}$ $(d)\ \frac{2}{5} \div \frac{3}{4} = \frac{8}{15}$

Now ask the pupil to attempt to discover how the quotient for example (a) might have been obtained by the use of some process with the fractions $\frac{2}{3}$ and $\frac{3}{4}$. Is it possible to get 8, the numerator of the quotient, by the use of any process with these numbers? Is it possible to get 9? Some pupils will quickly discover the desired rule. As soon as the pupils think they have discovered a rule to be used in solving further examples let them try it out to test its validity.

It is of interest to note that one of the great Hindu writers, Mahavira, stated the rule for the inversion of the divisor in the division of fractions as early as the middle of the ninth century. The device seems then to have dropped out of use until again adopted in Europe in the sixteenth century.

Division of Mixed Numbers. When the whole numbers are small, it is usually best to change mixed numbers to improper fractions before proceeding with the division. For example, $2\frac{2}{3} \div 4\frac{1}{5} = \frac{8}{3} \div \frac{21}{5} = \frac{8}{3} \times \frac{5}{21} = \frac{40}{63}$.

When the whole numbers are large, it is usually best to

first multiply both dividend and divisor by the least common multiple of the denominators of the fractions. For example, if we wish to divide $39\frac{1}{5}$ by $46\frac{3}{4}$ we first multiply both of these mixed numbers by 20, the least common multiple of 4 and 5. $39\frac{1}{5} \div 46\frac{3}{4} = 796 \div 935 = \frac{796}{935}$.

Complex Fractions. When indicating the process to be used in the solution of certain problems an expression sometimes results which is called a complex fraction. A complex fraction is merely an indicated division written in fractional form, such as the following:

$$\frac{3\frac{1}{2}}{5\frac{1}{3}}$$

This expression means the same as $3\frac{1}{2} \div 5\frac{1}{3}$. Similarly,

$$\frac{2\frac{1}{4}}{3\frac{1}{5}}$$

means the same as $2\frac{1}{4} \div 3\frac{1}{5}$. It is evident that complex fractions involve no new mathematics. To simplify a complex fraction, we perform the indicated division. Thus, $\frac{2\frac{1}{4}}{3\frac{1}{5}} = \frac{9}{4} \div \frac{16}{5} = \frac{9}{4} \times \frac{5}{16} = \frac{45}{64}$.

Complex fractions not infrequently arise when formulas are evaluated.

Unusually complex fractions such as $\dfrac{34\frac{1}{5} + 17\frac{1}{2} + 25\frac{1}{4}}{18\frac{1}{6} + 20\frac{1}{4} + 17\frac{1}{3}}$ should be omitted.

Attention should again be directed to the fact that the fractions actually in common use in practical life are very simple and the pupil should be able to perform mentally the operations involving these fractions. Short methods commonly used in dealing with common fractions are discussed on pages 198-212 and may be considered at this time.

The various operations with fractions should be taught with only as much theory and detailed explanation as seems necessary to a clear understanding of the subject. Pupils should not be required to memorize and repeat explanations. Teachers should understand the reasoning underlying the processes but it is not desirable that they attempt to force it upon immature pupils.

A Fraction as an Indicated Division. It is frequently necessary to change common fractions to decimals, as explained on pages 230-231. It is, therefore, necessary that the pupil be able to recognize a fraction as an indicated division. It is also necessary that the pupil get this concept of a fraction in order that he may clearly comprehend the reasons which underlie cancellation. It is not wise to introduce this conception of a fraction until most of the processes with fractions have been studied. The pupil will not have difficulty with this conception of a fraction unless too elaborate an explanation is attempted.

The pupil knows that $\frac{1}{2}$ of 8 involves the division of 8 into 2 equal parts. Similarly, $\frac{1}{3}$ of 6 means the division of 6 into 3 equal parts, and $\frac{1}{5}$ of 10 means the division of 10 into 5 equal parts. From such examples as the above the pupil is led to see the truth of the following.

$\frac{1}{4}$ of $8 = \frac{8}{4}$. This is the same as $8 \div 4$.
$\frac{1}{3}$ of $6 = \frac{6}{3}$. This is the same as $6 \div 3$.
$\frac{1}{5}$ of $10 = \frac{10}{5}$. This is the same as $10 \div 5$.

The pupil is then led to see that any fraction may be regarded as an indicated division. Thus, $\frac{3}{5}$ may be regarded as $3 \div 5$, and $\frac{7}{8}$ as $7 \div 8$.

A Fraction as a Ratio. After the ratio idea is understood the pupil may be gradually led to the ratio conception of a fraction.

However, the concept ratio is more abstract than that

of the fractional part of an object, and the pupil does not usually get the ratio concept of the fraction as readily as he gets the other concepts. The ratio idea involved is usually perceived by the pupil after sufficient emphasis has been placed on such problems as the following:

John had 4 marbles and lost 2. What part of his marbles had he left?

Fred needs 15 cents to buy a certain toy. He has 10 cents. What part of the money that he needs has he now?

SUMMARY OF THE FUNDAMENTAL PRINCIPLES OF FRACTIONS

Exceptional pupils in the grades will be interested to consider the five fundamental principles of common fractions, and to seek illustrations for each principle.

1. Multiplying the numerator of a fraction by a number, multiplies the fraction by that number.
2. Dividing the numerator of a fraction by a number, divides the fraction by that number.
3. Multiplying the denominator of a fraction by a number, divides the fraction by that number.
4. Dividing the denominator of a fraction by a number, multiplies the fraction by that number.
5. Multiplying or dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction.

After these principles are understood the pupils should be able to answer such questions as the following:

In what two ways may a fraction be multiplied?

In what two ways may a fraction be divided?

What operations may be performed on a fraction without changing the value of the fraction?

The History of Fractions. The history of the development of fractions shows that modifications of the treatment

of fractions were brought about slowly. The oldest known mathematical records indicate that fractions were extensively used at an early date. The systems, however, were cumbersome and the methods peculiar and complicated. This may partly account for the traditional difficulty of fractions.

Ahmes, an Egyptian scribe, probably lived about 1550 B.C. Evidence seems to indicate that he copied an earlier mathematical treatise. The original manuscript written by Ahmes is now in the British museum. In this manuscript the unit fraction is treated extensively. Ahmes used a special symbol to represent the fraction $\frac{2}{3}$, but he expressed other fractions by writing the denominator and placing over it a dot or a symbol called *ro*. Whenever a fractional value occurred that could not be thus expressed, it was written as the sum of two or more unit fractions. Thus, $\frac{2}{9}$ was written as $\frac{1}{6}+\frac{1}{18}$. $\frac{2}{97}$ was written as $\frac{1}{56}+\frac{1}{679}+\frac{1}{776}$.

The Babylonians used fractions having a denominator of 60 or 360. From this practice sexagesimal fractions developed. It is probable that 60 was chosen because its large number of integral divisors, 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30 greatly simplified work with certain fractions. The Babylonians also used certain unit fractions. Scientific use of sexagesimal fractions was made by Hypsicles in the second century, B.C. Ptolemy, a great astronomer of the second century, extended the use of these fractions.

The early Greeks used unit fractions and indicated them by simply writing the denominator with a double accent. Later they used ordinary fractions, writing the numerator once with a single accent and the denominator twice with a double accent. Thus, using our symbols, $\frac{9}{15}$ would have been written as $9'\ 15''\ 15''$. The Romans used duodecimal fractions; that is, fractions having 12 for a denominator. The inch and the ancient ounce were outgrowths of this. The Hindus used the general fractions and indicated them

by writing the numerator above the denominator without a line separating them. Thus $\frac{2}{3}$ was written $\overset{2}{3}$. The Arabs introduced the separating line. In an old Hindu arithmetic, the *Liliwati*, the multiplication of fractions is indicated by writing the fractions consecutively without any symbol between them. Addition of a fraction to an integer was shown by writing the fraction beneath the integer, and subtraction was shown in the same way with a dot prefixed to the fraction. Thus $4+\frac{2}{3}$ was written $\overset{4}{2}3$ and $4-\frac{3}{4}$, as $\overset{4}{3}4$. The problem, "Tell me, dear woman, quickly, how much a fifth, a quarter, a half, and a sixth make when added together," appears solved as follows:

1	1	1	1	67
5	4	2	6	60

That the subject of common fractions was a difficult one for the race is indicated by the fact that unit fractions were used for thousands of years. During the Middle Ages the term "common" was used to distinguish these fractions from sexagesimal fractions. Common fractions are called "vulgar" fractions in England.

Since the Arab invasion of Europe there has been but little change in the formal representation of fractions. Many of the rules found in *Liliwati* of the Hindus remain practically unchanged today. But fractions have increased vastly in importance; they are a common possession of the people; they are frequently needed in our everyday transactions. Textbooks give more space to them than formerly and the problems are more real and the instruction more skillful.

Problems Involving Fractions. All textbooks contain concrete problems in fractions. The problems differ greatly in statement, but an analysis will reveal the fact that they may be grouped under one of the following types:

Type I. Given a fractional relation and a number. Required to find a part of the number. For example, A boy earns \$12 a month and saves $\frac{3}{4}$ of the amount. How much does he save?

Type II. Given a fractional relation and a part of a required number. Required to find the number. For example, A boy saves \$9 a month which is $\frac{3}{4}$ of the amount that he earns. How much does he earn?

Type III. Given a number and a part of a number. Required to find the fractional relation. For example, A boy earns \$12 a month and saves \$9. What fractional part of his earnings does he save?

Type IV. Given the fractional relation and the number increased by a part of itself. Required to find the number. For example, A boy who sold a dog for \$12 gained $\frac{1}{3}$ of the cost. What was the cost?

Type V. Given the fractional relation and the number decreased by part of itself. Required to find the number. For example, A boy who sold a dog for \$12 lost $\frac{1}{3}$ of the cost. What was the cost?

The relationship of these types is seen by such a consideration as the following.

Type I.	$\frac{3}{4}$ of \$12 = ?	Two factors are given. Required to find the product.
Type II.	$\frac{3}{4}$ of ? = \$9	The product and one factor are given. Required to find the other factor.
Type III.	? of \$12 = \$9	The product and one factor are given. Required to find the other factor.

Type IV. $\frac{3}{3}+\frac{1}{3}=\frac{4}{3}$. $\$12\times\frac{3}{4}=?$ Two factors given to find product.

Type V. $\frac{3}{3}-\frac{1}{3}=\frac{2}{3}$. $\$12\times\frac{3}{2}=?$ Two factors given to find product.

A study of these types indicates that concrete problems in fractions involve only a few fundamental processes.

SUGGESTED READINGS

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CHAPTER XV

DECIMAL FRACTIONS

Origin of Decimal Fractions. There are two theories to account for the origin of decimal fractions. The advocates of the first theory, believe that decimal fractions were devised as a shorthand method for expressing certain common fractions. They assert that since the base of our number system is ten, someone devised or invented decimal fractions as a transition from common fractions. If this theory is correct, and if we follow the historical development of the subject in our teaching, the relationship to common fractions should be especially emphasized. This is the method followed by some textbooks.

Those who advocate the second theory maintain that decimal fractions were developed directly from the decimal scale for integers. Since numbers diminish in value from left to right in a tenfold ratio, because of place value, it would be natural to extend the number scale to the right of units place. Such an extension of the scale would give rise to the decimal fraction. Consider the number 111. The figure at the right indicates, by virtue of place value, one-tenth of the value indicated by the figure immediately to its left, and 0.01 of that indicated by the second figure to its left. If this same scale is extended to the right of the units place, we have the decimal fraction. This is the theory which is now quite generally held. If decimals originated in this manner, the subject is intimately related to integers and only indirectly related to common fractions.

A rational explanation of the processes involved in decimal fractions may be based upon either of these theories.

Decimal fractions were introduced so gradually that their origin cannot be ascribed to any one person. About 1585, a book was written by Stevin in which he attempted to show their great practical value. He says, "It is certain that, if the nature of man remains in the future as it is now, he will not always neglect so great an advantage." The world is not always quick to take advantage of a great improvement, as was evidenced by the tardiness in the adoption of the Gregorian calendar, and today by the opposition to the metric system. It was not until a century and a half after the time of Stevin that decimal fractions were generally taught. Some of the common fractions of the fifteenth century were so unwieldy that a change was imperative. One author used the fraction $34\frac{8}{8}\frac{0}{1}\frac{7}{0}\frac{3}{2}$.

Early users of decimals used a variety of methods. At present, either the point (period) or comma are in general use although there is yet no uniformity of practice in this matter. In the United States, *three and twenty-five hundredths* is expressed as 3.25. In England it is usually written and printed 3·25. In the Continental countries it may be written 3, 25, or 3^{25} or 3_{25}. It may even be written 3 25, merely a space being used to indicate the separation between the groups of figures. A 0 is sometimes written to the left of the decimal point to direct attention to the point; thus, 0.7 for .7.

Reading and Writing Decimal Fractions. After the conception of the decimal fraction has been made clear, practice should be given in the reading and writing of decimals. Teachers will find this work facilitated, if they require the pupils to memorize the orders of the decimal scale and the position of each order with reference to the decimal point. For example, the pupil should know that thousandths is the third decimal order, millionths the sixth,

and hundred-thousandths the fifth. If asked to write four hundred twenty-three millionths, the pupil should think at once that since millionths is the sixth order, and since there are three digits in four hundred twenty-three, he must insert three zeros. He begins, therefore, at the decimal point and writes three zeros, then the figures 4, 2, and 3. The pupil who writes decimal fractions from left to right does not need to count the digits after he has written the expression in order to insure the correctness of his work. We do not count the digits after we have written an integer to insure correctness, and there is no reason why we should do so in writing decimal fractions. Such work as the following is of value: How would you write twenty-four ten-thousandths? Answer. Decimal point, two zeros, 2, 4. When pupils are required to write decimal fractions from dictation, the enunciation should be very distinct, and the pupil should image the decimal completely before writing it. He should then write it continuously from left to right.

To read a pure decimal: Read as in whole numbers, then state the name of the decimal order of the figure at the right. Thus, .047 is read forty-seven thousandths.

To read mixed decimals: Read the integral part, then the decimal part, joining the two parts by *and*. Thus, 253.047 is read two hundred fifty-three *and* forty-seven thousandths.

Mixed decimals may also be read as improper fractions. Thus, 2.47 may be read as two hundred forty-seven hundredths.

Hundredths is often read as per cent. Thus, .72 may be read 72 per cent; 3.75 may be read three hundred seventy-five per cent.

There are certain types of decimal fractions that are frequently read incorrectly. Illustrations of these types follow:

2400.0006 should be read, twenty-four hundred *and* six ten-thousandths; .2406 should be read twenty-four hundred six ten-thousandths; $.0\frac{2}{3}$ should be read, two-thirds of a tenth, or two-thirds tenths; $.00\frac{2}{3}$ should be read, two-thirds of a hundredth, or two-thirds hundredths; $2.0\frac{2}{3}$ should be read, two *and* two-thirds of a tenth; $0.2\frac{2}{3}$ should be read two and two-thirds tenths; $2.2\frac{2}{3}$ should be read, two *and* two and two-thirds tenths.

Attention should be directed to the fact that in reading mixed decimals the word *and* should be used after reading the integral part.

During the seventh or eighth school year the attention of the pupil should be directed to the manner in which the practical man often reads decimals. For example, 37.462 would be read, thirty-seven, point, four, six, two.

Comparative Values. Pupils should be able to recognize comparative values in decimal fractions. Ask the pupils how much length is represented by each figure in .546 miles? How much money is represented by each figure in $.394? Pupils should realize that 0.1 is greater than .098 and that .01 is greater than .00998. Ask the pupil to state the decimal fraction of two digits that is nearest in value to $.57\dot{4}\dot{3}$, or to .03875. It is not wise to emphasize the decimal orders beyond the sixths, and most of the emphasis should be placed upon the first three orders.

Annexing Zeros. The effect of annexing zeros to the right of a decimal fraction should be understood. Ask the pupil to write .14; this is the same as .1+.04. If he should write a 3 to the right of the 4, the expression would be .143. How much has been added? Evidently, .003 has been added. If a 2 had been written instead of a 3, evidently .002 would have been added. If a zero had been written, evidently no thousandths would have been added, or, in other words, the value of the original expression would not have been changed.

The effect of annexing zeros to the right of a decimal

fraction may be explained also by showing that the annexing of zeros is equivalent to multiplying both numerator and denominator by the same power of 10. For example, $.14 = \frac{14}{100} = \frac{140}{1000} = \frac{1400}{10000} = .1400$.

Addition and Subtraction. Addition and subtraction of decimal fractions offer no difficulties except those previously met in the same processes with integers. Care should be taken that units of the same order are placed in the same vertical column so that the decimal points stand in a line. The decimal point in the result is then placed directly under the line of decimal points.

Example:	*Add*	*Subtract*
	17.326	43.826
	0.0437	27.438
	9.801	16.388
	0.42	
	27.5907	

Multiplication. The pupil knows that multiplication by an integer is but a short method for finding the sum of a number of equal addends. For example, the product of 7 and 38 may be found by using 38 as an addend 7 times. Similarly the product of 7 and 4.36 may be found by using 4.36 as an addend 7 times. This the pupil knows how to do. It is evident then that the multiplication of a decimal fraction by an integer must give a product with the same number of decimal places as there are in the multiplicand. The average pupil has no difficulty in determining the name or denomination of the product when asked to make any of the following multiplications: $7 \times \$3$; 7×3 miles; 7×3 pounds; 7×3 cents. He should be led to conclude the name or denomination of the product in such examples as the following: 7×3 hundredths; 7×3 thousandths.

The pupil is soon led to conclude that the product must

be hundredths or thousandths if the multiplicand is hundredths or thousandths. The first examples given to the pupil should contain no carrying over the decimal point.

Two methods are in general use for explaining the placing of the decimal point when a decimal is multiplied by a decimal. These methods are based upon the two conceptions of the origin of the decimal fractions.

By the first of these methods, the position of the decimal point in the product is determined by first reducing the decimals to common fractions, then making the multiplication, then expressing the product as a decimal fraction. The pupil knows the correctness of the procedure, and after working several examples by this method and comparing the number of decimal places in the product and in the two factors in each example, he soon discovers the principle and formulates the rule.

Thus: $.3 \times .15 = \frac{3}{10} \times \frac{15}{100} = \frac{45}{1000} = .045$

$.04 \times .007 = \frac{4}{100} \times \frac{7}{1000} = \frac{28}{100000} = .00028$

$4.27 \times .005 = \frac{427}{100} \times \frac{5}{1000} = \frac{2135}{100000} = .02135$

The second method is based on the theory that decimal fractions originated from an extension of the number scale to the right of the units order. Before taking up the explanation by this method the effect of moving the decimal point a given number of places to the right or the left should be thoroughly understood. Ask the pupil to compare the values of the following numbers: 100, 10.0, 1.00, 0.1, and to state the effect of moving the decimal point to the left. Use also 11111, 1111.1, 111.11, 11.111, and 1.1111. Apply the principle to the following and see if it is true: 428, 42.8, 4.28, .428, .0428. Similarly, teach the effect of moving the decimal point to the right. The pupil should also be led to discover, or review, the principle that when one of two factors is multiplied by any number and the

other factor is divided by this same number, the product of the two factors is not changed.

Examples:

$7\times4=\frac{7}{10}\times4\times10=28$ (Here one factor is divided by 10 and the other is multiplied by 10).

$5\times9=\frac{5}{17}\times9\times17=45$ (Here one factor is divided by 17 and the other is multiplied by 17).

In the first example above, 10 is introduced as a factor in the numerator and also in the denominator; hence we introduce $\frac{10}{10}$, or 1.

If the pupil is required to find the product of $4\times.07$ he may use the same principle as when he is required to find the product of $4\times\$7$ or 4×7 books. He knows that 4×7 hundredths is 28 hundredths, or he may write this as .28. If the example requires the product of $.4\times.07$ he may use the principles referred to above. Thus, he may multiply the .4 by 10 and divide the .07 by 10 without changing the value of the product. Hence, $.4\times.07=4\times.007=.028$. Similarly, $.03\times.005=3\times.00005=.00015$. (In this case one factor is multiplied by 100 and the other factor is divided by 100.) Similarly, $32.54\times.005=3254\times.00005=.16270$.

After the pupil has worked several examples he should, under the direction of the teacher, formulate his own rule for future use, and he should see that whatever operation is necessary to make the multiplier an integer should be performed upon it, and the inverse operation should be performed upon the multiplicand. If the multiplier is multiplied by 10, 100, or 1000, the multiplicand should be divided by 10, 100, or 1000, respectively, in order that the value of the product may be unchanged.

Multiplication of decimals may be explained also in the following manner. Required to multiply .043 by .05. If we multiply .043 by 5, the result is .215, but this example differs from the given example only in that the multiplier is 100 times as large as the given multiplier (since $5=100$

$\times .05$). Therefore the product is 100 times too large, and the correct product is $\frac{1}{100}$ of .215, or .00215.

Whatever method is used should appear to the pupil as rational, and, after a sufficient number of examples have been worked, he should formulate the rule that in order to multiply two decimal fractions we proceed as in multiplication of integers, and then point off in the product as many decimal places as there are in both the multiplicand and the multiplier. If the product does not contain as many figures as there are in the multiplicand and multiplier together, the necessary zeros must be prefixed to the product.

After the rule for multiplication has been rationally developed, the pupil should use it instead of developing the principle each time he wishes to find the product of two decimal fractions. The first essential is to make the process seem reasonable to the pupil, and the second is to acquire facility in the application of the process.

Such drill exercises as the following will be found valuable after the pupil has learned how to determine the placing of the decimal point in the multiplication of decimal fractions.

Do not work these examples but state the number of decimal places in the product in each case.

a. $.042 \times .017$	*e.* 0.749×3.056
b. $3.14 \times .526$	*f.* 8.0329×7.14
c. $.049 \times 8.73$	*g.* $.3207 \times .054$
d. $92. \times 4.806$	*h.* 9.308×17.0256

The figures of the product for each of the following examples are correct and in the correct order, but the decimal point has not been placed. Do not work the examples but read the correct products.

a.	$.073 \times 2.4 = 1752$	*e.*	$.0472 \times 36.9 = 174168$
b.	$4.036 \times 2.38 = 960568$	*f.*	$.572 \times 3.81 = 217932$
c.	$7.42 \times .308 = 228536$	*g.*	$.0043 \times 7.02 = 3096$
d.	$.028 \times .0407 = 11396$	*h.*	$.0504 \times .0713 = 359352$

Division. Division of decimals may be explained by first reducing the decimal fractions to common fractions, and, after performing the division, expressing the quotient by decimal notation.

Examples:

$.009 \div .03 = \frac{9}{1000} \div \frac{3}{100} = \frac{3}{10} = .3$

$.0045 \div .00005 = \frac{45}{10000} \div \frac{5}{100000} = 90$

After working several examples, the rule which expresses the relation between the number of decimal places in dividend, divisor, and quotient should be stated.

Most authors and teachers explain division of decimal fractions by use of the principle that multiplying or dividing both dividend and divisor by the same number does not change the value of the quotient. This principle is readily understood from a few examples. Thus, the quotient of $30 \div 6$ is not changed if both the 30 and the 6 are multiplied or divided by the same number.

$$30 \div 6 = (10 \times 30) \div (10 \times 6)$$

If the pupil is required to work the example $.06 \div 3$, he knows the quotient just as he knows the quotient if asked to divide \$6 or 6 books by 3. If required to divide .06 by .3 he may use the principle above. Thus, $.06 \div .3 = .6 \div 3 = .2$. (Both dividend and divisor were multiplied by 10.) Similarly $.00015 \div .005 = .15 \div 5 = .03$. (Here both dividend and divisor were multiplied by 1000.)

The pupil should be led to see that we first multiply both

dividend and divisor by whatever number will make the divisor an integer and then perform the division.

After a sufficient number of examples have been worked by the above method, the pupil should formulate a rule for division of decimal fractions. Divide as in integers and point off in the quotient a number of decimal places equal to the excess of the number of decimal places in the dividend over the number of decimal places in the divisor.

This rule may also be deduced from the fact that division is the reverse of multiplication, and the dividend is the product of the divisor and quotient. Since the number of decimal places in the product equals the number in both multiplicand and multiplier, the number of decimal places in the quotient must equal the number in the dividend minus the number in the divisor. We may, therefore, divide as with whole numbers and then so place the decimal point that this principle will not be violated.

A study made by Monroe[1] seems to indicate that a variety of devices are used by adults in determining the placing of the decimal point, and that the same persons sometimes use different devices in placing the decimal point in different types of examples.

In division of decimal fractions the quotient should be written above the dividend in such a manner that the decimal point of the quotient will be directly over the decimal point of the dividend.

Thus:
$$8\overline{)43.864}\quad 5.483$$

Indicated multiplication and division containing decimals may be rewritten with the divisors changed to integers.

[1] *Elementary School Journal*—Volume 18, pp. 287-293.

Thus:

$$\frac{.47\times12.25\times1.03}{.23\times2.05\times6}=\frac{47\times1225\times1.03}{23\times205\times6}$$

Both numerator and denominator are multiplied by 100×100.

Such drill exercises as the following will be found especially valuable in division of decimals.

Do not work the examples below, but state how many decimal places there will be in the quotient in each example.

a. $7.44\div6$
b. $87.6\div4$
c. $.001376\div.043$
d. $.0216\div7.2$
e. $.1575\div.045$
f. $.0428\div.07$
g. $1.9084\div7.34$
h. $.121706\div.302$
i. $.4344\div.6$
j. $3.9852\div.0738$

The figures in the following quotients are correct and in the correct order. Do not work the examples, but read the correct quotient for each example.

a. $157.25\div37=425$
b. $.0002172\div.00362=6$
c. $6.25\div2.5=25$
d. $.1728\div1.2=144$
e. $92.56\div13=712$
f. $9.375\div.25=375$
g. $.032375\div3.7=875$
h. $.1242\div4.6=27$

Approximate Results. In certain work in science, and in business activities, operations are often performed involving decimal fractions, when the result is desired correct to only a few decimal places. Perfect measurements are not possible in science, and results beyond three decimal places are very seldom required.

If a measurement is correct to only three decimal places, any computation in which it is directly involved can not be correct to more than three decimal places. In a com-

mercial transaction such a result as $43.492736 would be considered as $43.49. It is always a waste of time to carry out results to a greater degree of accuracy than the data upon which the results are based or the situation demands. It is usually not necessary to carry computation with United States money beyond two decimal places. An approximation which is correct to two decimal places cannot differ from the number approximated by more than .005 or $\frac{1}{200}$, and an approximation correct to three decimal places cannot differ from the number approximated by more than .0005 or $\frac{1}{2000}$. Decimal fractions which are equivalent to two places differ from each other by less than .01 and those which are equivalent to three decimal places differ from each other by less than .001.

The processes of multiplication and division involving decimal fractions may frequently be abridged by eliminating all unnecessary work. There is sometimes an advantage in multiplication in beginning with the highest instead of the lowest order of units of the multiplier.

In contracted multiplication and division, considerable practice is necessary to give one confidence in deciding correctly which is the farthest figure to the right that will affect the final result. Contracted methods are not as generally used in the schools as their importance would justify. The European schools use them more extensively.

Reduction of Decimal to Common Fractions. To reduce a decimal fraction to a common fraction, omit the decimal point, write the denominator of the decimal, and then reduce the common fraction to its lowest terms.

Thus: $.08 = \frac{8}{100} = \frac{2}{25}$.

This exercise will, in the case of some decimals involve complex fractions. Thus:

$$(a)\ .03\tfrac{1}{3} = \frac{3\frac{1}{3}}{100} = \frac{\frac{10}{3}}{100} = \frac{10}{300} = \frac{1}{30} \qquad (b)\ .027\tfrac{1}{2} = \frac{\frac{55}{2}}{1000} = \frac{55}{2000} = \frac{11}{400}$$

Reduction of Common to Decimal Fractions. The reduction of a common fraction to a decimal fraction may be explained in two ways.

First Method. A common fraction may be regarded as an indicated division. $\frac{2}{5}$ may be regarded as $2 \div 5$. Since we cannot take $\frac{1}{5}$ of 2 units exactly, we reduce the 2 units to the next lower denomination, thus making 20 tenths (2.0). $\frac{1}{5}$ of 20 tenths is 4 tenths, or .4. Therefore $\frac{2}{5} = .4$.

Reduce $\frac{2}{7}$ to a decimal fraction. Note that $\frac{2}{7}$ of 1 unit equals $\frac{1}{7}$ of 2 units. We cannot take $\frac{1}{7}$ of 2 units, so we reduce the 2 units to lower denomination and get 20 tenths, or 200 hundredths, or 2000 thousandths, etc.; $\frac{1}{7}$ of 2000 thousandths, equals $285\frac{5}{7}$ thousandths, or $.285\frac{5}{7}$.

This process may be performed briefly by placing a decimal point after the numerator and dividing by the denominator.

Second Method. Perform that operation upon the fraction that will make the denominator a power of 10, but will not change the value of the fraction. Then omit the denominator and express it by the position of the decimal point.

Thus: $\frac{3}{8} = \frac{3 \times 125}{8 \times 125} = \frac{375}{1000} = .375$ $\frac{4}{125} = \frac{4 \times 8}{125 \times 8} = \frac{32}{1000} = .032$

$$\frac{7}{16} = \frac{7 \times 625}{16 \times 625} = \frac{4375}{10000} = .4375$$

The first of these methods is easier for the pupil in most cases. The second method is sometimes briefer, but it requires more skill.

A common fraction cannot always be reduced to a pure decimal. Thus $\frac{1}{3} = 0.3\frac{1}{3} = 0.33\frac{1}{3} = 0.333\frac{1}{3}$. No common fraction can be reduced to a pure decimal if the denominator when the fraction is expressed in lowest terms, contains other prime factors than 2 and 5. This is true since 10

is the product of 2 and 5, and any power of 10 is the product of an equal number of 2's and 5's. In reducing such common fractions as $\frac{1}{3}$ or $\frac{1}{7}$ to decimal fractions the division may be carried as far as is necessary and then the common fraction or the plus sign may be written. Thus, $\frac{1}{7} = .14285\frac{5}{7}$, or .14285+.

Circulating Decimals. The reduction of certain common fractions to decimal fractions lead to the discovery of circulating decimals. Circulates are an outgrowth of the Arabic system of notation. The subject is very interesting, but it is more properly treated under infinite series in algebra than in arithmetic, and its discussion is omitted here for that reason.[1]

Aliquot Parts. The pupil should know the commonly used aliquot parts of 10, 100, and 1000 and should learn their decimal equivalents. The equivalents for $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, and for $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{5}{6}$ should be memorized.

Sequence of Common and Decimal Fractions. Decimal fractions are taught in most schools of the United States in the fifth or sixth grades. Special emphasis is usually placed upon the subject in the sixth grade, and percentage furnishes an opportunity to review decimals in the seventh and eighth grades. The sequence of common and decimal fractions is not yet settled. The general practice in the United States is to teach the simpler common fractions, such as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ informally in the early grades, and to treat the entire subject of common fractions more or less formally in the fifth grade. Decimal fractions are not often taught, except as they are involved in the writing of United States money, until the fifth or sixth grade. No one seriously advocates the complete development of common

[1] Teachers who are interested in the subject of circulating decimals will find it treated in the following references: Brooks, *Philosophy of Arithmetic*, pp. 460-485; McClellan and Dewey, *The Psychology of Number*, pp. 267-271; Showalter, *Arithmetical Solution Book*, pp. 49-55; Wentworth, *Practical Arithmetic*, pp. 354-356.

fractions before any consideration is given to the subject of decimal fractions.

The argument usually advanced by those who advocate the teaching of decimal fractions before common fractions is that the decimal fraction logically comes first since it is a natural outgrowth of our Arabic system of notation. Those who believe that common fractions should be taught first maintain that this sequence is in harmony with the historical development of the subjects, and that the concept of the common fractions is simpler than that of the decimal fractions. It is further urged that pupils do not grasp the importance of decimal fractions and the simplifications which they effect until they have worked with common fractions with unlike denominators.

Problems Involving Decimals. All of the problems of decimal fractions, as of all other subjects, should be as practical as possible. They should have a direct and intimate relation to the daily activities in which decimal fractions are so extensively used, and which come within the experiences of the pupil. Pupils should be urged to bring to class practical problems involving the use of decimal fractions.

Problems involving money, distance, time tables, speed and time; rainfall, temperature, crop yields, etc., are often of interest to pupils and furnish a good opportunity for computation in decimal fractions.

It has been asserted that the history of arithmetic shows a slow but well marked growth towards the decimal idea. The general adoption of decimal fractions has had a marked effect upon several other topics of arithmetic. Decimal fractions have greatly lessened the importance of the subjects of greatest common divisor and least common multiple and have simplified many of the calculations of science.

The fundamental operations with numbers expressed decimally are usually more easily performed than when the

numbers are expressed as common fractions. Moreover, numbers expressed decimally are much easier to compare than when expressed as common fractions. For example, if one basket-ball team played 11 games and won 8 and another played 13 games and won 10, it is not evident which team made the better record, but if it is stated that one team won .727 of the games played and the other won .769 of its games, it is at once evident which team made the better record.

The importance of decimal fractions is recognized more today than ever before and the subject is one of the most important in arithmetic.

SUGGESTED READINGS

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CHAPTER XVI

DENOMINATE NUMBERS

To some extent the development of a people can be judged by the systems of weights and measures in use. As science and industry develop, the units are refined and made more accurate. One writer says that the early units of weights and measures were born of necessity and developed by chance.

The story of the development of the various systems of weights and measures is a fascinating one. Many of the early units were based on some part of the human body, on some human activity, such as marching, or on the length or weight of some kind of grain.

Many of the names of our units bear mute evidence of their origin. In some cases the original meaning of the term has been changed. The wheat grain was used by the Egyptians, Romans, English, and by some other nations as the basic unit of weight. During the reign of Henry VII it was decreed that "one sterling (or penny) shall be the weight of thirty-two corris (grains) of wheat that grew in the midst of the ear, according to the old laws of the land." The early Hindus used the *gerah* which was the equivalent in weight of twelve grains of barley. The seed of a creeping plant called the rati was frequently used as the alternative to the grain of barley. The *carat*, a bean of an Abyssinian tree, was also used as a measure of weight by the Hindus and by some other nations when expressing the weight of precious metals.

The word *avoirdupois* appeared in early English as *haverdupois* and meant "goods of weight." The term was

introduced into England during the early fourteenth century. In England the avoirdupois pound contained 7,680 grains and the troy pound 5,760 grains.

The avoirdupois pound was formerly used for weighing meats only. Later it came into general use for the weighing of all common materials. During the seventeenth century troy weight gradually became restricted to use in weighing precious metals.

The long ton, which is extensively used in weighing coal at the mines and in the customs houses of the United States, had an interesting origin. The English *stone* was 14 pounds. Eight stones or 112 pounds was the integral number of stones next larger in weight than one hundred pounds and hence was called the *hundredweight*. The ton of twenty hundred-weight contained, therefore, 2,240 pounds and this is our long ton. The *pennyweight* was originally the weight of a penny.

The *cubit* was generally used by the Egyptians, Babylonians, and Hebrews as the length of the forearm from the elbow to the tip of the middle finger. The *inch* was defined by the early English as "the length of three barleycorns taken from the middle of the ear, well dried and placed end to end." The *inch* was defined by some as the length of the thumb from the first joint to the tip of the nail. The *yard* was decreed by Henry I to be the length of his arm. It is said that Edward III had the length of his own arm registered in a bar of metal and this was divided into three equal parts and each of these into twelve equal parts.

The *span* was defined as the distance between the tips of the thumb and little finger when the fingers are extended. The *hand*, now taken as four inches, was defined as the width of the hand. The *pace* refers to the length of the step. The *fathom* was the distance between the finger tips when the arms were extended at the sides. The *league* was the distance a man could see. The *furlong* was defined

as the distance oxen could plow without resting; the *mile* was a thousand paces — *millia passuum.*

Some measures of value must have been developed at an early date. Until some common measure of value was agreed upon it was not possible to engage in trade upon an equitable basis. The very necessities of exchange, where the wealth or property of some men consisted of furs, of others of cattle, of still others of tea, ornaments, tools, or weapons, required a unit of value. Occasionally this medium or standard of value was purely imaginary. John Stuart Mill[1] informs us that certain African tribes calculate the value of things in a sort of money of account, called *macutes*. One object might be worth ten macutes, another twenty, and so on. But macutes have no real existence; they are imaginary conventional units used for comparing the value of one object with that of another.

Money the Common Medium of Exchange. It was not always possible to exchange one object of value for another object of value. Distance must have made it impossible many times to make the transfer on the spot. A medium of exchange, a currency, became a necessity. Usually gold or silver served as money. The reason for this is simple. As soon as people were assured of the necessities of life, they naturally sought the more precious and the more ornamental things, like gold, silver, and precious stones. These are valuable because they are rare, imperishable, and easily portable. Gold and silver became the basis of exchange among many nations and tribes.

Money became both a unit of measure and an equivalent of value. Because it was a measurer and because objects varied in value, divisions and subdivisions of the unit were necessary.

It must be remembered that money was first weighed,

[1] *Political Economy*, chapter 5, p. 11.

not counted. Down to the time of medieval England coins were used as weights. This accounts for the English pound as a unit of weight. The Roman *pound* or *libra* was determined by a balance called the libra. Thus a libra of gold could be converted into a number of coins, or so many coins equaled a libra of gold. The standard libra varied with different emperors. In medieval England a pound troy came to mean the weight of 240 pennies or denarii.

Although a troy pound is supposed to consist of 5,760 grains and an avoirdupois pound of 7,000 grains, so that 7,000 grains troy equal 1 pound avoirdupois, the common grain is no longer used as the base unit. About the middle of the nineteenth century the British government adopted the length of a second's pendulum as the invariable unit of length. Of the 39.1393 parts into which it was divided the British yard contained 36. Each of these divisions is a British inch. The distilled water which fills a cubic inch always weighs the same amount, under the same conditions. This weight is divided into 252.422 equal parts; the British pound avoirdupois contains 7,000 of these parts.

Again, it is possible to give the quantity of distilled water which weighs exactly 10 of these pounds avoirdupois. This quantity of water always fills the same space. The space is the "content" of the British gallon, the unit of capacity. Our units of capacity and weight are clearly connected with and rest upon the firm basis of the "oscillation" unit of length.

The *quart* is one quarter of a gallon. The word *gill* comes from the Latin, *gilla*, a drinking glass. The word *peck* probably comes from the word "pack" meaning a certain load.

Uniformity in Measures. Wherever civilization has progressed and society has become more complex, varying and inexact units of measure have been found unsuited to the needs of men. Progress in civilization always neces-

sitates greater uniformity of the units of measure, and greater exactness in the units employed.

When the grain was defined to be "the weight of a grain of wheat taken from the middle of the ear and well dried" it was a variable unit and all those units of weight which were defined in terms of the grain were also variable. Our present system of weights and measures has one of the requisites of an ideal system — its units are accurately defined and are invariable, but it is not an ideal system when judged by the other requisite — the scale for ascending and descending reduction should be uniform in a given table. The metric system has both of these requisites and is therefore superior to the English system.

The necessity of adopting standards in the United States has been accompanied with a variety of problems. The different states have not had uniform systems of weights and measures. To obviate this difficulty there has been an insistent demand for the reformation of the weights and measures, and for the establishment and enforcement of a national system of standards.

The recognition of the English origin and variability of colonial measures was responsible for the statement in the Articles of Confederation that: "The United States in Congress assembled shall also have the sole and exclusive right and power of regulating the value of alloy and coin struck by their own authority or by that of the respective states, and of fixing the standards of weight and measures throughout the country."

The Constitution conferred upon Congress the power to coin money, regulate the value thereof and of foreign coin, and fix the standard of weights and measures.

After careful consideration the following were adopted: The yard of 36 inches, the avoirdupois pound of 7,000 grains, the gallon of 231 cubic inches, and the bushel of 2,150.42 cubic inches. The standard yard was the thirty-six inches

comprised between the twenty-seventh and sixty-third inches of a brass bar prepared for the court survey in London. The avoirdupois pound equals 1.215 pounds troy, which is the relative equivalent between these weights in England. The units of measure represent the wine gallon of 231 cubic inches and the Winchester bushel. These units were adopted in 1836, and the secretary of the treasury was directed to supply a uniform set to the governor of each state. By an act of 1893, the yard was defined as $\frac{3600}{3937}$ of a meter. Having thus provided the means of uniformity, Congress delegated the responsibility of enforcing it to the separate states. From time to time Congress has enacted laws governing the use and standardization of weights and measures. In 1901, the National Bureau of Standards was established.

The metric system was legalized in the United States in 1866, and is now extensively used in most scientific and in many commercial transactions. The movement to substitute it for the English system is not powerful enough to warrant the hope of its early adoption.

There is, however, a growing sentiment in favor of greater simplicity and uniformity as to the method involved in selling those commodities that are measured by the quantity or bulk. This sentiment favors the selling of such products and articles by weight. By the law of 1866, a bushel of wheat weighs sixty pounds, of corn or rye fifty-six pounds, of barley forty-eight pounds, of oats thirty pounds. Duties imposed upon these products are based upon weights rather than bulk measure. Of course, it would be false to assume that each of these weights equals a volume equivalent to the standard bushel 2,150.42 cubic inches. These weights represent the average obtained for these products in different localities for a number of years.

It is said that the early German traders introduced into England a coin which was called *easterling* and from this

the word *sterling* came. *Farthing* means a fourth of a penny.

The origin of the dollar mark as used in the United States is still somewhat in dispute, but the evidence seems to indicate that it originated from the Spanish mill dollar. Some have contended that the dollar mark originated from a combination of the letters *U* and *S*.

The Teaching of Denominate Numbers. Much of the material formerly included in the tables of weights and measures has been eliminated. The movement for greater simplicity in arithmetic is very evident in denominate numbers. The modern tendency is to teach only those tables that are generally employed by the majority of people and to leave the technical tables of the physician, the druggist, the jeweler, and the surgeon, to be learned by those who will have occasion to use them.

The standard for the selection of the materials to be taught in denominate numbers should be social usage. Only those tables should be taught which are extensively used in the ordinary pursuits of daily life. Pupils should be confronted with real arithmetical situations. To require all pupils to learn the tables of troy and apothecaries' weights and surveyor's measure is to force all to study that which only a few will ever need to know.

The entire subject of denominate numbers was formerly taught as a separate topic. The modern practice is to teach the various tables as the need for the tables arises. The various units of measure should be extensively used as necessity arises in the teaching of integers, common and decimal fractions. By the time the pupil comes to a study of the complete tables, he should have had occasion to study most of the more common units in natural arithmetical situations. The presentation of the complete tables should come relatively late in the course.

The pint and quart may be introduced to advantage when teaching multiplication and division by twos. The foot

and the yard may be introduced at the time the table of threes is studied. The quart and the gallon should be introduced with the fours; the nickel and the cent with the fives; the day and the week with the sevens; the quart and the peck with the eights.

Too often the units of weights and measures are only words, and remain such to the pupil because of the improper teaching of denominate numbers. The pupils should be permitted and encouraged to see, handle, and use the various units whenever this is possible. The foot should recall to the pupil a definite length and the pound a definite weight. By handling the various units of measure the pupil can make parts of his own tables. He should discover for himself that two pints make a quart and four quarts make a gallon.

During the first four school years the work in denominate numbers should be quite informal, and the pupil should become thoroughly familiar with all of the common measures.

Reductions of Denominate Numbers. No new mathematical principles are involved in the reduction of denominate numbers. The earlier arithmetics greatly overemphasized the importance of multiplication and division as applied to denominate numbers. Such operations are not frequent outside the school. Moreover, there are few arithmetical situations which require the addition or subtraction of units of several denominations. The value 12 feet 9 inches is often expressed as $12\frac{3}{4}$ feet or as 12.75 feet rather than by use of both feet and inches.

The earlier texts included many problems such as the following: Reduce 746,832 square inches to units of higher denominations. Reduce 3 miles, 4 rods, 2 yards, 2 feet, 7 inches to inches.

Such problems are not related to genuine situations and should be omitted.

Pupils should be asked to name the proper unit of measure

for expressing various measurements such as, the length of a pencil, the area of a farm, the distance between two cities, the size of a field, the weight of a bushel of grain, and the length of a room. Ask the pupils to name several things which the grocer sells by the pound, the ounce, the bushel. Ask them to name some things which the hardware dealer sells by the foot, by the pound. List several articles sold by the pint, the quart, the gallon.

Ask pupils to estimate numerous heights, lengths, widths, areas, and weights and then check their estimates.

The boy and the girl from the farm will be interested to learn such facts as the following:

1 bushel = 1.25 cubic feet; 1 cubic foot = .8 bushel; 1 heaped bushel = 1.6 cubic feet; 43,560 square feet = 1 acre; 1 ton of settled timothy hay in the mow = 450 cubic feet; 1 ton of settled clover hay in the mow = 550 cubic feet.

The girl should be taught that:

A measuring cup full of flour, butter, granulated sugar, or water weighs about half a pound.

Recently published arithmetics contain many problems such as the following:

5 yd. = ____ ft.	48 in. = ____ ft.	64 oz. = ____ lb.
4 lb. = ____ oz.	15 ft. = ____ yd.	4 ft. = ____ in.

Such problems are practical and concise. The pupil must think in order to solve them. Many teachers place on the blackboard a drill table such as the following:

320	144	2000	5280
231	$31\frac{1}{2}$	1728	9
160	196	12	2150.4
3	16	27	$5\frac{1}{2}$

Pupils are asked to go to the board and read, "320 rods = 1 mile; 231 cubic inches = 1 gallon." The reverse of this

table may also be used to advantage in certain drill exercises. Sometimes pupils confuse weight and volume. Weighing the same volume of different things will assist them to avoid this confusion.

Many pupils see no relationship between the tables for linear and for square measure. So teach these tables that the pupil will see that square measure is derived directly from linear measure.

The subject of denominate numbers may be made very interesting and practical. Much of the work should be oral. Long and involved problems in the addition, subtraction, multiplication, and division of denominate numbers should be omitted. Problems involving changes through more than two denominations seldom occur in business. Be sure so to teach the subje : that the units used have a real and definite meaning to the pupils. So many teachers make the mistake of talking about the various units of measure, but not presenting them to the pupils in such a way that they become to them real, concrete, definite, and usable.

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CHAPTER XVII

THE METRIC SYSTEM

Essentials of an Ideal System. An ideal system of weights and measures must have two characteristics: The units must be accurately defined and invariable, and there must be a uniform ratio between consecutive units in any given table. The early systems of weights and measures had neither of these characteristics. The necessity for accurately defined units became more imperative as commerce and industry developed, and eventually the units were defined with scientific accuracy.

Science improved the old system of weights and measures by imparting to its units the exactness and uniformity which is the first essential, but it did not establish the simple scale of relationships between the units, which is the second essential.

The lack of uniformity in the ratio between consecutive units of the English system may be seen by a consideration of the following:

12 inches = 1 foot	16 ounces = 1 pound
3 feet = 1 yard	100 pounds = 1 hundred-weight
$5\frac{1}{2}$ yards ($16\frac{1}{2}$ feet) = 1 rod	20 hundred-weight = 1 ton
320 rods (5,280 feet) = 1 mile	

The simplicity of the metric tables can be judged from the following:

10 mm. = 1 cm.	10 m. = 1 dm.
10 cm. = 1 dm.	10 dm. = 1 hm.
10 dm. = 1 m.	10 hm. = 1 km.

Not only is there a lack of uniformity of ratio between the consecutive units throughout most of the tables of the English system, but the same raio is rarely maintained for three consecutive units. In linear measure the numbers expressing the relationship between consecutive units are 12, 3, $5\frac{1}{2}$, and 320. In surface measure the ratios are, 144, 9, $30\frac{1}{4}$, 160, and 640.

It is evident that the English system of weights and measures is lacking in the second essential of an ideal system — a uniform scale of relationship between the consecutive units of a given table.

Origin of the Metric System. The metric system is the result of an attempt by the people of France to devise an ideal system of weights and measures,— a system that would have both of the essentials that have been mentioned. The metric system is the result of extensive scientific study and is not a chance product. It was devised for a definite purpose. The system was the product of the minds of five of the most eminent mathematicians of France. It was devised during the last decade of the eighteenth century. Its history is very interesting and will repay study.

The name of the system is derived from the French word *metre*, meaning "to measure." The meter was defined to be one ten-millionth of the distance from the equator to the pole, and this was taken as the standard of linear measure. All other units of the system were correlated directly with the meter. It was later discovered that a slight mistake was made in computing the length of the meter, but this error does not render the system less valuable.

Present Status of the Metric System. The enlightened judgment of educated people pronounces the system to be the best the world has yet seen. It is endorsed with enthu-

siasm by those who have studied it, and many predict that it will ultimately become the universal system among civilized nations. The persistence of custom will delay its ultimate triumph for many years. Since its adoption in France, it has spread with great rapidity. Its use is today either obligatory or permissive in every civilized country of the world. It is in general use among all civilized nations except England and the United States. Instruction in the metric system is obligatory in all English schools. It has been the legal system in the United States since 1866. It is used by the United States Army and Navy; foreign postal rates are based on metric units; our yard is defined in terms of the meter; and the metric system is used in Porto Rico and in the Philippines.

The Future of the Metric System. We live in an age of international activities and relationships. Commerce demands that articles for export and for import shall be measured by units that are understood among civilized nations. The complex business activities of the day make the use of a simple system imperative. One advocate of the metric system calls attention to the fact that "there is no nation today having a decimal system of weights and measures or currency where there is a suggestion of a change, and no nation not having a decimal system where there is not constant agitation in favor of a decimal system. Whenever a civilized nation has changed its units the change has always been to a decimal system."[1] Another ardent advocate of the metric system states the following: "Considered merely as a labor-saving machine it is a new power offered to man incomparably greater than that which he has acquired by the agency which he has given to steam. It is in design the greatest invention of human ingenuity

[1] Hon. James E. Southard, "*School, Science and Mathematics*," 1905. pp. 653-657.

since that of printing."[2] The metric system is not only easy to learn but it is easy to apply because of its simplicity. There are two objections advanced to the general adoption of the metric system in the United States. The first is a real objection, but it is not so formidable as is supposed; i. e., the cost of changing tools to correspond to the metric units and the training of the laborer to think and work in terms of these units. A sudden change to the metric system would be attended with confusion, but no such change will take place, because the adoption of the new units will be made gradually. Our money system is now on a decimal basis. In recent years several of the large manufacturing concerns in the United States have adopted the metric units for exclusive use in shops, and the reports indicate little or no confusion as a result of the change. The second objection urged against the adoption of the metric system is the fact that in the matter of weights and measures we are strongly bound by tradition. The chain of tradition is weakening, however, and the metric system is gaining ardent advocates at the expense of the English system.

It is desirable that teachers should appreciate the merits of the metric system. No teacher can afford to be ignorant of a system that has been so generally adopted by civilized nations and is the basis for all scientific work. A well-educated person today should know the metric units in common use. The English system should receive in arithmetic much greater emphasis than the metric system, however, for it is the system that most pupils will use in their daily activities.

The Metric System in School. If the course of study provides for the teaching of the metric system, it is better to teach the system in the seventh or eighth grade than

[2] Hallock and Wade, "*Evolution of Weights and Measures and the Metric System*," pp. 116-118.

earlier, because it can be closly correlated with the work in the sciences which are usually emphasized in these grades. The system should be taught only when some use of it is to be made.

How to Teach the Metric System. The metric units in common use should be actually in hand. The best results cannot be secured unless the pupils see and handle most of the commonly used measures and they should be first studied as metric units, without regard to their nearest English equivalents. Pupils should be trained to think in terms of metric units. Many efficient teachers introduce the metric system by showing the similarity of the system to the tables for United States money. The power to estimate accurately in terms of these units may be developed through drill. Ask the pupils to hold two fingers, one, two, or three centimeters apart. Estimate the linear and surface measure of numerous common objects. Most of the work in the metric system should be oral. After the units are well understood and some facility has been acquired in their use, a brief study of the equivalents of the English system may be made. (A table of equivalents is given at the close of this chapter.)

The abbreviations for the metric units are not uniform. Square meter is written as qm, or m^2, or M. Each compound name in the metric system is accented on the first syllable, thus: kil′ometer; mil′limeter.

The prefixes must be thoroughly learned if one is to master the metric system. The pupil must know that "kilo" always means 1,000, whether it is prefixed to the unit, meter, liter, or gram. A kilogram is 1,000 grams; a kilometer is 1,000 meters; a kiloliter is 1,000 liters. Similarly "milli" when prefixed to any units means .001 of that unit. Thus, millimeter means .001 of a meter. The following mnemonics may help the beginner to associate the new terms with names which are familiar. The word "meter"

is familiar to the pupil in such terms as "water meter," "gas meter," and "cyclometer." The term always means "a measure." *Deci*, meaning .1, may be associated with a dime, 0.1 of a dollar. *Centi*, meaning .01, may be associated with cent, .01 of a dollar. *Milli*, meaning .001, may be associated with mill, .001 of a dollar. If the term *dekagram*, a ten-sided figure, is familiar to the pupil, he will more easily remember that *deka* means ten. *Myria* is easily remembered as a large multiplier by associating it with the word *myriad*.

The prefixes should be drilled upon until they are mastered, and the pupil is able to translate back and forth in the table. The prefixes which indicate multiples of the unit are derived from the Greek, and those which indicate the decimal divisions of the unit are from the Latin.

The remarkable simplicity of the metric system and the ease with which reductions may be made should be emphasized. For example, compare the ease with which 487.26 kilometers may be reduced to meters with a similar reduction in the English system. Reductions in the metric system involve only the moving of the decimal point; thus, the subject gives an opportunity to review decimals. Since 1,000 liters equal 1 cubic meter, the capacity of bins and tanks may be computed with much greater ease than by use of the English units.

The five-cent piece may be used to illustrate some of the metric units. Its diameter is 2 centimeters; its thickness is 2 millimeters, and its weight is 5 grams. The definitions of meter, liter, and gram should be learned.

A meter is .0000001 of the distance from the equator to the pole. It is the unit of length and is the base of the entire metric system.

A liter (pronounced "lēter") is the capacity of a cube .1 meter on an edge. It is the unit of capacity.

A gram is the weight of a cube of distilled water .01 of a meter on an edge. It is the unit of weight.

Prefix	Meaning	Illustration	Meaning
myria	10,000	myriameter	10,000 meters
kilo	1000	kilogram	1000 grams
hekto	100	hektogram	100 grams
deka	10	dekameter	10 meters
deci	0.1	deciliter	0.1 liter
centi	0.01	centimeter	0.01 meter
milli	0.001	millimeter	0.001 meter
mikro	0.000001	mikrometer	0.000001 meter

TABLE OF EQUIVALENTS

A meter	= 39.37 inches = $3\frac{1}{4}$ ft. nearly = 3 ft. $3\frac{3}{8}$ inches, nearly
A liter	= 1 quart, nearly
A kilogram	= 2.2 pounds, nearly
A gram	= 15.43 grains
A hektare	= 2.47 acres

The meter is used for dry goods, merchandise, engineering, construction, and other purposes where the yard and foot are used in the English system.

The centimeter and millimeter are used instead of the inch and its fractions in machine construction and similar work.

The centimeter is used in expressing sizes of paper and books.

Any quantity consisting of several denominations may be treated as an integer and a decimal, the decimal point separating the unit and its divisions. For example, 1,420.25 meters is not read 1 kilometer, 4 hektometers, 2 dekameters, 2 decimeters, and 5 centimeters, but is read 1,420 meters and 25 centimeters, just as $1,420.25 is read 1,420 dollars and 25 cents.

UNIT	WHERE EMPLOYED
Centimeter	Commerce, industry, and science
Decimeter	Commerce, industry, and science
Dekameter	Surveying

Unit	Where Employed
Gram	Jewelry and most drugs and letters
Hektare	Measuring land
Hektoliter	Dry measure
Hektometer	Artillery
Kilogram	Groceries and general weights
Kilometer	Distances in general
Liter	Liquid measure
Megameter (1,000,000 meters)	Astronomy
Metric ton	Large weights, such as hay and coal
Mikron (1 millionth of a meter)	Metrology, spectroscopy, microscopy
Millimeter	Commerce, industry, and science
Millimikron	Metrology, spectroscopy, microscopy
Myriameter	Geography
Stere	Measuring wood

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CHAPTER XVIII

PERCENTAGE

Percentage Is a Language Lesson. The subject of percentage is essentially a language lesson; no new mathematical principles are involved in any of its applications. The business man doesn't say that his business has increased one-fourth, he says that it has increased 25 per cent. When the child knows one-fourth, he knows 25 per cent, all but the name. The problem is to teach the pupil to think familiar ideas in a new language. Efficient teachers bear in mind the fact that most of the difficulties involved in percentage are language difficulties. Pupils should not feel that a change in terminology involves new mathematical processes.

Some textbooks treat the subject of percentage as if it involved new mathematical principles. In such books the subject is treated under various cases, rules, and formulas. The best teachers of arithmetic today do not present the subject in this way. McClellan and Dewey state that the teaching of percentage by cases, rules, and formulas is a mistake from both the theoretical and the practical points of view. It is a mistake on the theoretical side, "because it asserts or assumes a new phase in the development of number; on the practical side, because it substitutes a system of mechanical rules for the intelligent application of a few simple principles with which the student is perfectly familiar."[1] In actual business practice, problems are not classified under their rules and cases. Moreover, the

[1] McClellan and Dewey, *The Psychology of Number*, p. 279.

terms *base*, *rate*, and *percentage* are not often used in business with the same meanings that were formerly taught in the schools.

Relation between Percentage and Fractions. Percentage is a continuation of fractions. It is a special case of the subject and it may be made to afford excellent practice in enlarging the ideas of fractions and in securing greater facility in using them. The following illustration will serve to show the language change in the transition from fractions to percentage. A man had 700 chickens and sold 1 out of every 2, or 1 out of every 10, or 1 out of every 50, or 1 out of every 100, or 1 out of every 350, how many were sold? The above problem would be classified under the subject of fractions. Since 1 out of 2 was sold, the number sold was $\frac{1}{2}$ of 700; or since 1 out of 10 was sold the number sold was $\frac{1}{10}$ of 700; or since 1 out of 100 was sold, the number sold was $\frac{1}{100}$ of 700.

If we agree that the word *per* shall mean *out of* when used in this connection, the above statements would be: 1 per 2, or 1 per 10, or 1 per 100. If we substitute the Latin word *decem* for *ten*, our statement becomes 1 per *decem;* if we substitute the Latin word *centum* for *hundred*, our statement becomes 1 *per centum*. If we abbreviate the word *centum* by cutting off the last two letters we have *1 per cent*. We frequently abbreviate words in this way in arithmetic; for example, we write *int.* for *interest*, and *fract.* for *fraction*. Therefore, 1 per cent means 1 out of every 100. If a problem involving 1 out of every 100 is properly classified as a problem in fractions, it is certain that a problem involving 1 per cent is also a problem in fractions. Percentage came in as a separate topic about the beginning of the nineteenth century.

The fact that a quantity is measured off into hundredths instead of into any other possible number of parts appears to be no valid reason for considering percentage as a new

phase in the development of number. Is the process of computing by hundredths "to be broadly distinguished as a mental operation from the process of computing by eighths, or tenths, or twentieths, or fiftieths? If the difference between fractions and percentage is not a difference in logical or psychological processes, but chiefly a difference in handling number symbols, is it worth while to invest the subject with an air of mystery and to invent, for the edification of the pupil, from six to nine 'cases' with their corresponding rules and formulas?"[2]

One reason why percentage is so extensively used today may be readily seen by a consideration of the following: If you are told that one school basket-ball team won 7 out of 13 games, and that another team won 6 out of 11, you are not certain which team made the better record although you know that one team won $\frac{7}{13}$ and the other won $\frac{6}{11}$ of the games played. If, however, these fractions are expressed as per cents, and you are told that one team won 53.8 per cent and the other won 54.5 per cent of the games played, it is evident at once which team made the better record.

Applications of Percentage. Percentage admits of applications in many fields, and the extensive use of decimal fractions is increasing the number of problems to which it is applied. In the textbooks of a few decades ago, almost all problems in percentage had a financial basis, but today the best textbooks contain numerous problems involving percentage in which money considerations are in no way involved.

One of the chief difficulties which pupils encounter when studying the application of percentage is the fact that they do not get a clear comprehension of the terms used. Unless the pupil has a clear conception of the terms employed,

[2] McClellan and Dewey, *The Psychology of Number*, p. 280.

he cannot form accurate judgments and conclusions. Too often the terms used are but words, words, words, to the pupil. The various applications of percentage should be taught by correlating them as closely as possible with the actual arithmetical situations.

Anything that increases the interest of the pupil and helps him to associate the terms and the problems with actual situations in which they occur is worth while, unless an excessive amount of time is expended. Any material that will vitalize the work may be used to advantage by the teacher. The pupil observes business customs and procedures in his daily activities, and his observations should be encouraged and made as accurate as possible.

It is important that the pupil should have an understanding of his natural environment, and it is important that he should have an understanding of his social and business environment. An understanding of both natural and business environments is essential in the life work of a successful individual.

Whatever problems are used in percentage should be in harmony with actual business, industrial, and scientific practice. All problems should be adjusted to present-day activities. Dewey's criticism that much that is learned in school cannot be applied in daily life is especially pertinent here. "The fact that the arithmetic of business is the arithmetic of common sense should not for a moment be lost sight of in drilling classes in this branch of our schools."[3] The teacher should seek to train the pupil to grasp conditions and to develop the power to apply the processes to actual business and industrial situations. All applications of percentage not consistent with modern practice should be omitted. One of the chief objects should be to develop commercial and industrial efficiency and social insight.

[3] N. E. A. Committee Report on Business College Course, Vol. II., p. 2163.

Equivalents of Certain Common Fractions. The teacher should impress upon the mind of the pupil the fact that *per cent* is identical in meaning with *hundredths*. That 6 per cent of a quantity means 6 *one-hundredths* of it. $10\% = \frac{10}{100} = .10$; $100\% = \frac{100}{100} = 1.00$; $357\% = \frac{357}{100} = 3.57$; $\frac{3}{8}\% = \frac{3}{8}/100 = .00375$.

Problems involving the reduction of common fractions to per cent and of per cent to common fractions should be solved. Most of the work should be done orally, and the pupil should be familiar with the fractional equivalents for the most common per cents of business. These should be memorized and drilled upon. The following are very commonly used.

$6\frac{1}{4}\% = \frac{1}{16}$	$12\frac{1}{2}\% = \frac{1}{8}$	$25\% = \frac{1}{4}$	$62\frac{1}{2}\% = \frac{5}{8}$
$8\frac{1}{3}\% = \frac{1}{12}$	$16\frac{2}{3}\% = \frac{1}{6}$	$33\frac{1}{3}\% = \frac{1}{3}$	$66\frac{2}{3}\% = \frac{2}{3}$
$10\% = \frac{1}{10}$	$20\% = \frac{1}{5}$	$50\% = \frac{1}{2}$	$75\% = \frac{3}{4}$
		$37\frac{1}{2}\% = \frac{3}{8}$	$100\% = 1$

The equivalent for $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{14}$, and $\frac{1}{15}$ are not frequently used.

There should be drill also in expressing decimal fractions as per cent. For example: $.1 = 10\%$; $.13 = 13\%$; $.9 = 90\%$; $.004 = \frac{4}{10}\%$, or $.4\%$; $.0005 = \frac{5}{100}\%$, or 0.05%; $0.00625 = \frac{5}{8}\%$; $.00\frac{2}{3} = \frac{2}{3}\%$; $5.24 = 524\%$.

The pupil should see that the reduction of a decimal fraction to per cent involves simply the reduction of the decimal fraction to hundredths, and then instead of reading it *hundredths*, we call it *per cent*, which means *hundredths*.

Three Basic Type Problems of Percentage. In percentage and its application there are but three basic mathematical ideas involved, and all of these are familiar to the pupil before he begins the formal study of the subject. The most complex problems in percentage contain no new

mathematical principles. A pupil who has mastered the four fundamental operations with integers, and with common and decimal fractions, and who can reduce common fractions to decimal fractions, and vice versa, should have little or no difficulty with the mathematics of percentage. He may be unable to solve the problems, but his failure will be due to a failure to understand the terms employed and to think old ideas in terms of new symbols. Not the mathematics, but the language of percentage is the crux of the subject. Since only three basic mathematical ideas are involved in the problems of percentage, it is economical of time and effort for the teacher to emphasize these three ideas before proceeding far with the subject. The first two of these are much more commonly used than the third.

The first type problem is: *To find any per cent of a given quantity.* Find 7% of 45. The pupil knows that *per cent* means *hundredths*, and from his study of decimal fractions he knows how to find .07 of a given quantity.

$$7\% \text{ of } 45 = .07 \times 45 = 3.15$$
$$32\% \text{ of } 372 = .32 \times 372 = 119.04$$

To find any per cent of a quantity, multiply the given quantity by the given per cent expressed as hundredths. It is evident that a per cent of a given quantity may be regarded as the product of two factors.

When per cents greater than 100 per cent are involved, the meaning *out of 100* is not so clearly understood. The pupil will understand the meaning of 200 per cent, however, if the fact that per cent means hundredths is thoroughly understood. He will know that $200\% = \frac{200}{100} = 2$; $145\% = \frac{145}{100} = 1.45$. The pupil should be taught to express any per cent as a decimal.

Find $\frac{3}{8}\%$ of 416. $\frac{3}{8}\% = .00375$; therefore, $\frac{3}{8}\%$ of $416 = .00375 \times 416 = 1.56$. Or we may say, $\frac{3}{8}\% = \frac{\frac{3}{8}}{100} = \frac{3}{800}$; therefore, $\frac{3}{8}\%$ of $416 = \frac{3}{800}$ of $416 = 1.56$.

It is evident that this type involves no new mathematical knowledge. The symbols involved are new, but the principle is not new.

The second type problem is: *To find what per cent one number is of another.* For example: 4 is what per cent of 8?

In the preceding problems we were required to find a given per cent of a number. In this problem we are told that, if we take a certain per cent of 8, the result is 4.

In other words, we have given the product of two factors, also one of the factors to find the other factor. This involves no new principle. If we divide the product, 4, by the given factor, 8, the result is the other factor. Since the problem asks for the result in terms of per cent (hundredths), we express the result not as $\frac{1}{2}$, but as $\frac{50}{100}$ or 50%.

Similarly: 7 is what per cent of 17?

$? \times 17 = 7$. The unknown factor is found to be $\frac{7}{17}$, or $.41\frac{3}{17}$ or $41\frac{3}{17}\%$.

The third type of problem is: *To find a number when a certain per cent of it is known.*

For example: 40 per cent of a number is 10. What is the number?

The product of two factors is 10; one of the factors, .40, is known. To find the other factor, divide the product, 10, by the known factor, .40. The result, 25, is therefore, the other factor, or the required number. $.40 \times ? = 10$.

Some teachers prefer to explain these processes by the use of unitary analysis. The above explanations, however, are brief and are easily understood by the pupils; furthermore, they involve no new mathematical ideas and they are serviceable.

In many classes practically all drill work in percentage

involves per cents less than 100; most of the per cents are less than 50. Since many practical problems involve per cents larger than 100, pupils should receive adequate drill with such per cents. Problems involving gain in weight, size, and population may be used to advantage. Such questions as the following are suggested: Is your father more than 100 per cent older than you? Name some grain or flower that grows more than 100 per cent in one month. Is it possible to increase your savings more than 100 per cent?

The Solution of Problems. The pupil who has a thorough mastery of the preceding principles does not need to solve the problems of percentage by rules and cases. Sometimes he should use the purely fractional form, at other times the percentage form, and at still other times he will combine the fractional and the percentage forms. "Practice in the application of the principles should enable the pupil to use both forms with equal facility, and to determine in any given problem which of the forms will lead to the most elegant and concise solution." No set form should be required after the reasons for the various processes are understood, but the teacher should require the pupil to express himself in concise and accurate language. Too great insistence upon minute details in written work may disgust and discourage some pupils.

If the problem states that 25 per cent of a number is 18, the pupil should immediately see that the number is 4×18, or 72. If the problem requires $33\frac{1}{3}$ per cent of 480, the pupil should take $\frac{1}{3}$ of 480. Some object to the use of common fractions in problems which involve percentage. It must be remembered that percentage is studied in the grades because of its practical applications, and whenever possible the pupil should solve the problems of percentage in the manner in which these problems are solved in actual life out of school. No business man would solve the following problem by keeping it in its percentage form. Twenty-five

per cent of the cost of an article is \$14; find the cost. The practical man would immediately find the cost by finding $4 \times \$14$. "The arithmetic of business is the arithmetic of common sense."

A Source of Failure in Percentage. One of the most frequent sources of error in the study of percentage is due to the failure of the pupil to keep clearly in mind per cent of what. In all early work in percentage teachers should require pupils to name the quantity upon which the per cent is based.

Questions such as the following will help to impress this important point.

1. What is 25 per cent of 40?
2. 40 is 25 per cent of what number?
3. 40 is what per cent of 25?
4. 40 is 25 per cent greater than what number?
5. 25 is what per cent less than 40?
6. What number is 25 per cent less than 40?
7. 40 is 25 per cent less than what number?
8. 40 is what per cent greater than 25?

The following type of problem is especially good to impress the importance of keeping in mind the quantity that is to be made the basis in a problem involving percentage.

Water expands 10 per cent of its volume when it freezes. What per cent of its volume does ice contract when it melts?

The artificial problems found in many textbooks are of the same general type. A's money is 25 per cent of B's more than B's. B's money is what per cent of A's.

Solution: Since A's money is 25 per cent of B's more than B's, therefore, A's money = 125 per cent of B's money; therefore, 1 per cent of B's money = $\frac{1}{125}$ of A's

money; therefore, 100 per cent of B's money $= \frac{100}{125}$ of A's money, or $\frac{4}{5}$, or 80 per cent of A's money.

The problem may also be solved by using common fractions throughout.

The Types of Problems. Numerous problems which do not involve money should be given in percentage. The percentage of increase or decrease in population for various cities from 1910 to 1920; the percentage of games won or lost by the school teams; the percentage of attendance at school on a given day; the average heights of boys and of girls at different ages and the yearly percentage of increase; the average weight and strength at different ages; the per cent of pupils in each grade of the school based on the total school enrollment; the percentage of boys and girls in a given grade; the per cent of waste in cutting the largest possible circle out of a square; the per cent of decrease when various articles are dried or baked; the percentage of nutritive matter in various food articles; the per cent of yield of a given crop per acre as compared with other years; these suggest a few of the many problems not involving money that may be given to the pupils.

CHAPTER XIX

GAIN AND LOSS

The subject of gain and loss is one of the most important applications of percentage. It is closely connected with mercantile activities and the problems should be in harmony with business practice. The pupils should be informed in regard to genuine business situations and practices. The experiences of the pupils should be utilized as the basis for problems whenever this is possible. The mercantile and industrial interest of the community should be the source of numerous problems. In most of the problems involving gain and loss, which the business man is called upon to solve, the cost is known. Sometimes a merchant knows a possible selling price and must decide how much he can afford to pay to make a certain per cent. Some problems involving inverse cases may be properly introduced.

What Is Meant by the *Cost*. It is now the practice of many manufacturers to include under the term *cost* all the expenses commonly included in the production of an article. In efficient farm accounting the cost is usually understood to include all expenses involved.

There are many items of expense connected with an article before it is actually sold. The merchant and the manufacturer must pay out money for rent, insurance, taxes, advertising, wages, fuel, light, and transportation of goods. He must pay interest on his investment and he must make an allowance for depreciation. Although all these items of expense must be paid, it is not yet the general practice to include all of them under the term *cost*.

The meaning of the word *cost* must be determined from

the situation and from the business practice in the particular industry or business and in the community.

Computing the Per Cent of Gain and Loss. Many business men now compute the per cent of gain or loss on the selling price. This is a definite and a known amount; moreover the first cost represents only a part of the merchant's investment. If the merchant bases his per cent of gain on the original purchase price only, then a part of the money which he has invested is yielding no income. Whether the cost is computed on the purchase price, or on the purchase price increased by certain buying and selling expenses, or on the selling price, no confusion arises in a given case when the basis is stated.

If an article which cost \$3 is sold for \$4, what per cent of the cost is gained? What per cent of the selling price is gained?

The gain is \$1. The gain is, therefore, $\frac{1}{3}$ or $33\frac{1}{3}$ per cent of the cost.

The gain is \$1. The gain is, therefore, $\frac{1}{4}$ or 25 per cent of the selling price.

In general it may be said that a gain of $\frac{1}{3}$ of the cost is the same as $\frac{1}{4}$ of the selling price. This may be shown as follows:

Since the gain is $\frac{1}{3}$ of the cost, therefore $\frac{4}{3}$ of the cost is equal to the selling price, therefore $\frac{1}{3}$ of the cost is equal to $\frac{1}{4}$ of the selling price, therefore, $\frac{3}{3}$ of the cost is equal to $\frac{3}{4}$ of the selling price.

Since the cost is $\frac{3}{4}$ of the selling price, the remainder of the selling price, or $\frac{1}{4}$ of it, must be the gain.

Similarly, show that a gain of $\frac{1}{5}$ of the cost is the same as a gain of $\frac{1}{6}$ of the selling price.

No Rules or Cases Are Necessary. When teaching the applications of percentage, the teacher should be certain that all pupils have the proper understanding of the terms used. No formal rules or cases are necessary or desirable

in solving the problems in gain and loss. The problems in gain and loss may be grouped under five general headings. It is understood that the per cent of gain and loss may be based on either the cost or on the selling price.

I. Given the cost and the rate per cent of gain or loss, based on the cost, to find the gain or loss, and the selling price.

II. Given the cost and the gain or loss, to find the rate per cent of gain or loss, based on the cost, and the selling price.

III. Given the gain or loss, and the rate per cent of gain or loss, based on the cost, to find the cost.

IV. Given the selling price and the rate per cent of gain or loss, based on the cost, to find the cost.

V. Given the cost and the selling price to find the rate per cent of gain or loss, based on the cost; on the selling price.

The procedure should be suggested by the nature of the problems.

Many problems should be solved without the use of paper or pencil. For example:

(*a*) The cost is 8 cents and the selling price is 10 cents. Find the per cent of the cost which is gained.

(*b*) The cost is 12 cents and the selling price is 10 cents. Find the per cent of the cost which is lost.

In problem (*a*), the pupil should see at once that the gain is 2 cents and the per cent gained is whatever per cent 2 cents (the gain) is of 8 cents (the cost). The only mathematics involved in such problems is a subtraction and "to find what per cent one quantity is of another quantity."

Some Type Solutions. The following solutions are intended to be merely suggestive. It is not assumed that they are the only correct ones. Pupils should be encouraged

to use their initiative. Teachers and pupils are cautioned against inaccurate expression in this subject.[1]

1. Cost = \$420. Rate of gain = 20% of the cost. Find gain and selling price. 20% (or $\frac{1}{5}$) of \$420 = \$84 = gain.

Selling price = \$420 + \$84 = \$504.

It is evident that the selling price might have been determined by finding 120 per cent (or $\frac{6}{5}$) of \$420.

2. The cost is \$36. The loss is \$4. Find the selling price and the per cent of loss based on the cost.

The selling price is \$36 − \$4 = \$32.

The per cent of loss is whatever per cent \$4 (the loss) is of \$36 (the cost).

The loss is $\frac{1}{9}$ or $11\frac{1}{9}$ per cent of the cost.

3. The gain is \$81. The rate of gain is $12\frac{1}{2}$% of the cost. Find the cost and selling price.

$12\frac{1}{2}$% (or $\frac{1}{8}$) of cost = \$81.

Therefore 100% (or $\frac{8}{8}$) of cost = \$648.

Therefore selling price = \$648 + \$81 = \$729.

4. The selling price was \$560 and the rate of loss was 20 per cent of the cost. Find the loss and the cost.

Since the selling price was 20% of the cost less than the cost, it was 80% of the cost.

Therefore 80% of the cost = \$560.

Therefore 100% of the cost = \$700.

Therefore the loss = \$140.

The above problem may also be solved as follows:

Let c = the cost

$0.20c$ = the loss

$0.80c$ = selling price = \$560

therefore $c = \frac{\$560}{.80} = \$700.$

The introduction of the symbo c is a decided advantage in the solution of such problems, and more emphasis should be given to such solutions in the schools.

[1] See chapter on Accuracy, p. 106

5. The selling price was \$30; the cost was \$18. Find rate per cent of the selling price that was gained.

The gain was \$30 − \$18 = \$12.

The rate of gain based on the selling price is whatever per cent \$12 (the gain) is of \$30 (the selling price). \$12 is $\frac{4}{10}$, or 40 per cent, of \$30. The gain is, therefore, 40 per cent.

In every problem in gain and loss in which the per cent of gain or loss is given, three *per cents* (or fractions) may always be found.

The first of these is always known. It is this: 100% of the cost = the cost.

The second of these is always given in the problem.

The third may always be found by adding or subtracting the first two.

For example, in the second problem above, it is known that 100 per cent of the cost = cost.

20% of cost = loss.

80% of cost = selling price.

Instead of using the *per cent*, the equivalent fractions may be used.

For example: The selling price is \$390; the rate of gain is 30% of the cost; find the cost.

$\frac{10}{10}$ of the cost = the cost. (This is always known.)

$\frac{3}{10}$ of the cost = the gain. (The problem states this.)

$\frac{13}{10}$ of the cost = the selling price. (By combining the first two.)

Therefore, $\frac{13}{10}$ of the cost = \$390.

$\frac{10}{10}$ of the cost = $\frac{10}{13}$ of \$390 = \$300.

The above problem may be more briefly solved as follows:

c = the cost.

$1.30c$ = \$390.

$$c = \frac{\$390}{1.30} = \$300.$$ The cost is \$300.

A man sold two lots for \$150 each. He gained 25% of

the cost on the first and lost 25% of the cost on the second. Find his entire gain or loss in the transaction.

Solution:

125% of cost of first lot = selling price of first lot.

(Since it was sold at a gain of 25%.)

Therefore, 125% of cost of first lot = \$150.

100% of cost of first lot = \$120.

75% of cost of second lot = \$150. (Since it was sold for \$150 at a loss of 25%.)

100% of cost of second lot = \$200.

Cost of both lots was \$320. (\$120 + \$200.) Selling price of both was \$300.

The loss was \$20. The rate of loss was whatever per cent \$20 (the loss) is of \$320 (the cost). \$20 is $6\frac{1}{4}\%$ of \$320. The loss was, therefore, $6\frac{1}{4}\%$. Such problems as this frequently confuse pupils because they do not keep in mind the *per cent of what.*

I sold $\frac{2}{3}$ of an article for $\frac{3}{4}$ of the cost of the whole article. Find the rate per cent of gain or loss based on the cost.

Since $\frac{2}{3}$ of the article sold for $\frac{3}{4}$ of the cost of the whole, $\frac{1}{3}$ of the article sold for $\frac{1}{2}$ of $\frac{3}{4}$, or $\frac{3}{8}$ of the cost of the whole; therefore $\frac{3}{3}$ of the article sold for $\frac{9}{8}$ of the cost of the whole.

The gain was, therefore, $\frac{1}{8}$ of the cost, or $12\frac{1}{2}\%$.

The above solution may be considerably abridged by using *c* for *cost* and *s.p.* for *selling price.*

Thus: $\frac{2}{3}$ *s.p.* = $\frac{3}{4}c.$
$\frac{1}{3}$ *s.p.* = $\frac{3}{8}c.$
$\frac{3}{3}$ *s.p.* = $\frac{9}{8}c.$
Gain = $\frac{1}{8}c.$ = $12\frac{1}{2}\%$.

A careful study of the illustrative problems in gain and loss will corroborate the statement found on page 265 in

regard to the mathematics involved in the applications of percentage.

The teacher should supplement the textbook problems in gain and loss by using many of the problems given under the subject of fractions. Many teachers encourage the pupils to dramatize the various business situations in order to make them more real.

CHAPTER XX

DISCOUNT

The subject of discount is usually taught in the seventh and eighth grades. It is intimately related to business activities and is one of the most important applications of percentage. Discount is the deduction from the list price of an article. Discounts are usually expressed in per cents, though fractions are sometimes used, as $\frac{1}{3}$ *off* and $\frac{1}{2}$ *off*. The trend of business practice in recent years has been towards per cents that are easily reduced to common fractions, so that the discount can be computed easily. Thus, a discount of $16\frac{2}{3}$ per cent is easier to compute than one of 15 per cent. Discounts are sometimes quoted in the following manner: 20 and $\frac{1}{3}$ off. This means that a discount of 20 per cent is taken from the list price, and then a discount of $33\frac{1}{3}$ per cent ($\frac{1}{3}$) is taken from this remainder.

Reasons for Discounts. Among the most important reasons for granting discounts are: For cash; to avoid the frequent publication of large catalogues because of fluctuations in market prices; for large amounts purchased. Wholesalers and manufacturers usually publish a *list price* and then allow a certain per cent of discount *to the trade*. Goods listed but not subject to discount are marked *net*. A wholesale house usually has some such statement as the following upon its bills: Terms: "30 days, 2% off 10 days," which means that purchasers are entitled to a credit of 30 days, but the purchaser will be allowed 2 per cent discount if the bills are paid within 10 days. Terms of discount are often expressed as follows: $\frac{2}{10}$; $\frac{1}{30}$; $\frac{n}{60}$; meaning 2 per cent discount for 10 days, 1 per cent for 30 days, and no discount if the bill is not paid before 60 days. When

possible, teachers should consult trade journals and discount sheets which quote prices and discounts.

The subject of discount should be made real to the pupils rather than a technical discussion of business procedure. Urge pupils to bring to class advertisements which offer goods at discounts. Such advertisements may be found in the daily metropolitan papers, and in all communities sales are advertised at certain seasons. "After Christmas Sales" and "Pre-Inventory Sales" are common. Ask the pupils to list the probable reasons for discounts which are offered and require the pupils to compute the rate of discount or the amount of discount offered. In many communities discounts are given on electric light, gas, and telephone bills if the bills are paid before a certain date. Most teachers can obtain for class use some receipted bills on which discounts were allowed.

Successive Discounts. When more than one discount is given, the first discount is based on the list price, and the others are based on the successive remainders after the preceding discounts have been deducted. No two successive discounts are reckoned upon the same amount, hence successive discounts can never be added to find the equivalent single discount. Two successive discounts of 20 per cent and 10 per cent are not equivalent to a single discount of 20 per cent plus 10 per cent. In all problems involving discounts the pupil must keep in mind the basis upon which the discount is to be computed.

The idea is quite prevalent that successive discounts must be removed in the order in which they are quoted. The discounts may be removed in any order. The order does not affect the result. Successive discounts of 20 per cent, 10 per cent, and 5 per cent will produce the same net price whether removed in the order just stated or in any other order. The truth of this statement may be easily verified.

Assume any marked price, for example, $360. Suppose the discounts are 20, 10, and 5.

First Solution

$360 = list price
.20
$72.00, first discount
$360 − $72 = $288 = basis for second discount
.10
$28.80, second discount
$288 − $28.80 = $259.20 = basis for third discount
.05
$12.96, third discount
$259.20 − $12.96 = $246.24 = net price.

Second Solution

$360 = list price
.10
$36.00, first discount
$360 − $36 = $324 = basis for second discount
.05
$16.20, second discount
$324 − $16.20 = $307.80, basis for third discount
.20
$61.56, third discount
$307.80 − $61.56 = $246.24 = net price.

The net price is the same in each case.

It is apparent that the basis for the second discount might have been found by taking 80 per cent of $360, instead of first finding 20 per cent of $360 and subtracting this amount from $360. Similarly, the basis for the third discount might have been found by taking 90 per cent of

the basis for the second discount, and the net price might have been found by taking 95 per cent of this amount. In other words, we first find 80 per cent of the marked price, then 90 per cent of this amount, and finally 95 per cent of this amount. The above operations may be indicated as follows:

$.80 \times .90 \times .95 \times \$360 = \$246.24$. It is apparent that the result is not affected if the .90 and .80 are interchanged or if any other interchange of the .80, .90, and .95 is made.

The fact that a change in the order of successive discounts does not change the list price may be easily shown by use of algebraic symbols.

Finding Equivalent Discounts. It is well for the teacher to know the short method for finding the single discount equivalent to two successive discounts.

Example: Find the single discount equivalent to the successive discounts of 20 and 10.

A discount of 20 per cent leaves 80 per cent of the list price to be paid. A further discount of 10 per cent leaves 90 per cent of the 80 per cent of the list price, or it leaves 72 per cent of the list price to be paid. Since 72 per cent of the list price is to be paid, the two discounts of 20 per cent and 10 per cent must be equivalent to 28 per cent of the list price (100 per cent of list price $-$ 72 per cent of list price $=$ 28 per cent of list price).

This equivalent may readily be found as follows: First add the two discounts; then multiply them and take .01 of their product; subtract this result from their sum; the remainder will be the single discount equivalent to the two successive discounts.

Applying the rule just stated to the discounts, 20 per cent and 10 per cent, we have the following: $20 + 10 = 30$. $20 \times 10 = 200$. .01 of $200 = 2$. $30 - 2 = 28$. A discount of 28 per cent is equivalent to discounts of 20 per cent and 10 per cent.

Find the single discount equivalent to the successive discounts of 15 per cent and 5 per cent. The sum is 20 per cent. The product is 75. .01 of the product is .75. $20-.75=19.25$. Therefore, a discount of $19\frac{1}{4}$ per cent is equivalent to discounts of 15 per cent and 5 per cent.

If required to find the single discount equivalent to three successive discounts, first, find the single discount equivalent to two of the discounts and then find the single discount equivalent to this result and the third discount.

For example: Find the single discount equivalent to the discounts of 20 per cent, 10 per cent, and 5 per cent.

Discounts of 20 per cent, and 10 per cent are equivalent to a single discount of 28 per cent. Discounts of 28 per cent and 5 per cent are equivalent to a single discount of 31.6 per cent. Therefore, discounts of 20 per cent, 10 per cent, and 5 per cent are equivalent to a single discount of 31.6 per cent.

Similarly, discounts of 20 per cent, 30 per cent, and 10 per cent are equivalent to a single discount of 49.6 per cent.

Why Discounts Are Quoted Separately. Pupils sometimes inquire why a firm does not quote a single discount instead of quoting two or three successive discounts. Why shouldn't a firm quote a single discount of 31.6 per cent instead of three discounts of 20 per cent, 10 per cent, and 5 per cent? When the discounts are quoted separately, a purchaser knows what each discount is given for and he may take advantage of one or two of them, even though he may be unable to take advantage of the three. The last discount quoted may be *for cash*. The purchaser may be prepared to take advantage of the other discounts, but he may be unable to pay cash for his goods. If the three discounts were grouped as a single discount, a purchaser would not know the deductions to be allowed for specific reasons.

Illustrative Problems

The list price of some goods was \$540. The discounts were 15%, 10%, and 6%. Find the net price.

First Solution

.85×.90×.94 of \$540 = \$388.314 = net price.

Second Solution

Discounts of 15%, 10%, and 6% are equivalent to a single discount of 28.09%. A discount of 28.09% leaves 71.91% of the list price to be paid.

.7191 of \$540 = \$388.314 = net price.

A few indirect problems in commercial discount are usually given in textbooks. To illustrate:

What must be the marked price of goods to give discounts of 25%, 10%, and 10% and still realize \$243.

\$243 = net price.

Discounts of 25%, 10%, and 10% are equivalent to a single discount of 39.25%.

This leaves 60.75% of list price, which = net price.

Therefore, 60.75% of list price = \$243.

Therefore, list price = \$400.

The following is also a solution for this problem:

Let l = list price

Then .75×.90×90 of l = \$243.

$$\text{Therefore } l = \frac{\$243}{.75\times.90\times.90} = \$400$$

A merchant buys goods at a discount of 40% from the list price and sells at a discount of 30% from the list price; what per cent of the cost does he gain?

The cost is 60% of the list price.

The selling price is 70% of the list price.

The gain is 10% of the list price.

Since the per cent of gain is based on the cost, we must find what per cent 10 per cent of the list price is of 60 per cent of the list price (or the cost). It is $\frac{1}{6}$ of it, or $16\frac{2}{3}$ per cent of it. Therefore, the gain is $16\frac{2}{3}$ per cent of cost.

At what per cent above cost must goods be marked in order to give a discount of 20% on the marked price and still make a gain of 30% on the cost?

Since a discount of 20% is given, the selling price is 80% of the marked price.

Since a gain of 30% is made, the selling price is 130% of the cost.

Therefore, 80% of the marked price equals 130% of the cost.

1% of the marked price equals $\frac{13}{8}$% of the cost.

100% of the marked price equals $\frac{1300}{8}$% of the cost, or $162\frac{1}{2}$% of the cost. Therefore, the goods must be marked $62\frac{1}{2}$% above cost.

It should be noticed that no mathematics is involved in the above problems except the four fundamental operations and the three type problems of percentage previously referred to. The last problem solved involves the finding of a quantity (the list price) when a certain per cent of it is known.

Marked Goods. This is not a necessary part of a study of commercial discount or of gain and loss, but pupils will be much interested in the subject and a brief consideration may be given.

Merchants frequently indicate the cost price and the selling price on each article. In order to conceal these from the customer, the merchant usually resorts to some symbols as a private mark. Usually some word or phrase containing ten different letters is used as a *key* or *cipher*. These letters are used to represent the nine digits and zero.

An extra letter is used to prevent the repetition of a letter; this is called a *repeater*. The repeater prevents giving any clew to the private mark and it renders the deciphering more difficult. If the cost and selling price are both written on the same tag, the selling price is usually written below and the cost above a horizontal line, but these positions are sometimes reversed. Sometimes the cost price is written in cipher known only to the proprietor.

Suppose the key is *Pay us often*, and *x* is used as a repeater.

1	2	3	4	5	6	7	8	9	0	Repeater
p	*a*	*y*	*u*	*s*	*o*	*f*	*t*	*e*	*n*	*x*

An article which cost \$4.60 and is to sell at \$5.40 is marked

$$\frac{uon}{sun}$$

An article which cost \$5.56 and is to sell at \$6.50 is marked:

$$\frac{sxo}{osn}$$

How would the cost and the selling price above be indicated if the key were *Buy for Cash?*

The following list of key words and phrases may be used by the class. Select a few of the keys and ask the pupils to express various costs and selling prices by using them. Pupils will be interested to suggest other keys that may be used.

Importance	Charleston	Blacksmith
Republican	Buckingham	Gambolines
Authorizes	Harmonizes	Bridgepost
Let him work	Cumberland	Our last key
Hard moneys	Buy for cash	Cash profit
The big four	United cars	Black horse
Frank Smith	Don't be lazy	Now be quick
Now be sharp	He saw it run	You mark his
Market sign	Big factory	No suit case

CHAPTER XXI

SIMPLE INTEREST

Pupils sometimes fail to master the subject of interest, but the failure may usually be traced to the lack of an accurate understanding of the terms used and of inadequate acquaintance with business procedure, rather than to any mathematical difficulties involved. There is no problem in simple interest the solution of which requires a degree of mathematical knowledge not in the possession of the pupil who is prepared to begin a formal study of the subject. Simple interest is an easy application of percentage with time as an additional factor.

Many children know what interest is before they begin the study of the subject in the schools. The children of many cities and towns have deposits in savings banks, and they know that the bank pays them a small amount each year for each dollar that they have on deposit. Many children have heard their fathers speak of borrowing or lending money at a certain rate of interest. Some own Liberty Bonds. Or they may have read of money being borrowed to use in building the schoolhouse, or some other public building.

The pupils should understand that most successful merchants and manufacturers borrow money from the banks at certain seasons of the year. Many farmers borrow money and then repay it when their crops are sold. Some pupils have the idea that the borrowing of money is an indication of poverty. This idea should be corrected. Judicious borrowing by responsible people is to be commended.

Numerous definitions have been suggested for the term *interest*. Some textbooks define it as money paid or charged for the use of money. The statement that *interest is money rent*, is probably as serviceable in the grades as any definition that has been proposed.

The pupil should understand how men, when lending money, require a certain payment for the use of the money. They know that, if one rents a boat, he must pay a certain amount for the use of the boat. The boat, however, does not become his permanent property. One who rents a house or a farm pays a certain amount for the use of the house or the farm.

The teacher should make clear to the pupils how a man can afford to borrow a given sum for a year and at the end of the year pay back to the lender not only the amount originally borrowed, but an additional amount, which is called interest. The sum upon which interest is based is called *principal*, to distingish it from the interest which is of subordinate importance.

Legal Interest versus Usury. In the United States the legal rate of interest is fixed by law in several of the states. If one charges more than the legal rate it is called *usury*. The penalty for charging usury differs in various states. In some states the offender loses all of the interest; in others he loses both principal and interest; in others he may be fined or imprisoned. Some states provide no penalty for usury. In some states the usury laws have been repealed and the tendency is to allow an open market for money. Teachers may borrow a statute from a justice of the peace and read the law on interest and usury to the pupils.

The following table shows the present interest rates prevailing in the various states.

TABLE IX

States	Legal Rate	Rate Allowed by Contract	States	Legal Rate	Rate Allowed by Contract
Alabama.....	8%	8%	Montana....	8%	10%
Alaska.....	8	12	Nebraska....	7	10
Arizona.....	6	10	Nevada.....	7	7
Arkansas.....	6	10	N. Hamp....	6	6
California....	7	12	New Jersey..	6	6
Colorado.....	8	12	N. Mexico...	10	12
Connecticut..	6	6	New York...	6	6
Delaware....	6	6	No. Carolina.	6	6
Dist. of Col...	6	8	No. Dakota..	6	10
Florida.....	8	10	Ohio.....	6	8
Georgia.....	7	8	Oklahoma...	6	10
Hawaii.....	8	12	Oregon.....	6	10
Idaho.....	7	10	Pennsylvania.	6	6
Illinois.....	5	7	Porto Rico...	6	12
Indiana.....	6	8	Rhode Island.	6	Any
Iowa.....	6	8	So. Carolina..	7	8
Kansas.....	6	10	So. Dakota..	7	12
Kentucky....	6	6	Tennessee...	6	6
Louisiana....	5	8	Texas.....	6	10
Maine.....	6	12	Utah.....	8	12
Maryland....	6	6	Vermont....	6	..
Massachusetts	6	Any	Virginia.....	6	6
Michigan....	5	7	Washington..	6	12
Minnesota...	6	10	W. Virginia..	6	6
Mississippi...	6	8	Wisconsin...	6	10
Missouri.....	6	8	Wyoming....	8	12

The *legal rate* means that rate of interest which may be collected legally when the words *with interest* are included in a note but no rate of interest is specified. The

legal rate is also the rate which may be collected on a note without interest which is not paid when due. Such a note draws the legal rate of interest from the date of maturity until it is paid. It will be seen from the preceding table that in some states the legal rate is the same as the rate allowed by contract; in others it is less.

What Determines Interest Rates? The pupils should consider the various factors which determine interest rates. The relation of interest rates to the law of supply and demand for money should be pointed out. Other things being equal, interest rates are low when the people have much money to lend; rates are high when money is scarce and borrowers have trouble in getting all they need. Special attention should be directed to the security of the loan as a factor in determining interest rates. The United States Government can borrow large sums of money at relatively low rates of interest. An unstable government must pay high interest rates. It is the duty of the teacher to impress upon the pupils the fact that very high rates of interest are often synonymous with poor security.

The first thing to be investigated in making a loan is not the rate of interest that is charged, but the security of the loan. It would not be correct to say that high rates of interest are always directly associated with poor security.

Another factor which determines interest rates is the time for which the loan is made. The rate is usually lower upon a loan for a long period than for a short period. The amount of the loan is also a factor in determining interest rates. The rate for a small loan is often higher than for a large one.

Days of Grace. To protect borrowers the law in some states allows a short time, usually three days, called days of grace, beyond the time stated in the note, to make payment. Days of grace have been abolished in many of the states because the development of banking facilities has

rendered them unnecessary. The teacher should ascertain whether days of grace are still permitted in his state; if not permitted, they should be omitted from all problems, irrespective of the statement in the textbook.

Exact Interest. Since the year contains approximately $365\frac{1}{4}$ days, there is, strictly speaking, no such thing as exact interest. However, the name exact interest is applied to simple interest computed upon the exact number of days of each month and on the year of 365 days.

Exact interest is used by the United States Government, by some large banks, and in finding the interest on foreign money.

Exact interest for any number of days may be found by taking $\frac{72}{73}$ of the amount of the common interest. (360 days is $\frac{72}{73}$ of 365 days.) Common interest may be found from exact interest by increasing the exact interest by $\frac{1}{72}$ of itself. (365 days is $\frac{73}{72}$ of 360 days.)

Various Methods. Pupils should be taught only one method besides the general or direct method for computing interest. The attempt to develop mastery of several special methods usually results in confusion. It is better to secure a mastery of one good method than a partial mastery of several. There is some difference of opinion as to the best special method for computing interest. No one method is used exclusively by business men, who do not use interest tables. The practice varies. The tendency in recent years seems to be towards the bankers' method, six per cent, the aliquot part, the cancellation, and the formula methods, although a number of the others have commendable features. One who has frequent occasion to compute interest may become quite expert in the use of any of the special methods.

The Bankers' Method. The bankers' method is one of the short methods for computing interest. For example:

Find the interest on \$4,800 for 48 days at 6%. Solution:

Int. on \$4,800 for 60 days = \$48
Int. on \$4,800 for 30 days = \$24
Int. on \$4,800 for 6 days = \$4.80
Int. on \$4,800 for 18 days = \$14.40

Explanation: Since 60 days is $\frac{1}{6}$ of 360 days, therefore the interest at the rate of 6% for 1 year will be 1% for 60 days. We, therefore, can find the interest for 60 days by moving the decimal point in the principal two places to the left. This amount, \$48, is put down. We then try to find certain numbers which are simple fractional parts of 60. We find the interest for 30 days by taking $\frac{1}{2}$ of \$48; then for 6 days by taking $\frac{1}{5}$ of \$24, since 6 days = $\frac{1}{5}$ of 30 days. We then find the interest for 18 days — three times 6 days. Now 30 days + 18 days = 48 days. Therefore, the interest for 48 days is \$24 + \$14.40 = \$38.40.

If the rate is 5% we may first find the interest at 6% and then multiply this result by $\frac{5}{6}$. If the rate is 7% we may find the interest at 6%, then increase this by $\frac{1}{6}$ of itself. When finding interest it is usually not necessary to carry out the interest to more than three decimal places.

The Six Per Cent Method. The six per cent method is one of the shortest for finding the interest on \$1 for a given number of days. Since it is based on a year of 360 days, it is somewhat inexact. However, in many financial transactions the difference between the interest computed on the basis of a year of 360 days and that computed on the basis of 365 days is negligible.

The six per cent method is based on the following facts.

Interest on \$1 at 6% for 1 year (360 days) = \$.06
Interest on \$1 at 6% for 2 mo. ($\frac{1}{6}$ of 1 yr.) = \$.01

Interest on \$1 at 6% for 1 mo. ($\frac{1}{2}$ of 2 mo.) = \$.005
Interest on \$1 at 6% for 6 da. ($\frac{1}{5}$ of 1 mo.) = \$.001
Interest on \$1 at 6% for 1 da. ($\frac{1}{6}$ of 6 da.) = \$.000$\frac{1}{6}$

It is readily seen that the interest on \$1 at 6% for 1 month is $\frac{1}{2}$ cent. The interest on \$1 at 6% for any number of months is half as many cents as there are months. Thus the interest for 8 mo. is \$.04; for 7 mo. it is \$.035; for 11 mo. it is \$.055.

It is evident also that since the interest on \$1 at 6% for 1 day is $\frac{1}{6}$ of a mill, the interest on \$1 at 6% for any number of days will be $\frac{1}{6}$ as many mills as there are days. Thus, the interest on \$1 at 6% for 18 da. is \$.003; for 21 da. it is \$.0035.

The above facts enable us to determine readily the interest on \$1 at 6% for any number of months or days. Required to find the simple interest on \$480 for 7 mo. 18 da. at 6%.

The interest on \$1 at 6% for 7 mo. 18 da. is \$.038.

$\therefore$ the interest on \$480 at 6% for 7 mo. 18 da. is $480 \times \$.038 = \$18.24$.

If the rate were 4% we would take $\frac{4}{6}$ of the interest just found. If the rate were $3\frac{1}{2}$% we would divide the above interest by 6 (which would give the interest at 1%) and then multiply this result by $3\frac{1}{2}$.

Sometimes the six per cent method is long and cumbersome in application.

The Aliquot Parts Method. In computing interest by the *aliquot parts method*, the time is separated into two or more parts so that each part less than a year is a unit fraction of some preceding part. One example will make the method clear.

Required to find the simple interest on \$480 for 1 yr. 7 mo. 18 da. at 5%.

Five per cent of \$480 is \$24, interest for 1 year.

6 mo. $= \frac{1}{2}$ yr.
1 mo. $= \frac{1}{6}$ of 6 mo.
15 da. $= \frac{1}{2}$ mo.
3 da. $= \frac{1}{10}$ mo.

\$12. int. for 6 mo.
\$ 2. int. for 1 mo.
\$ 1. int. for 15 da.
\$ .20 int. for 3 da.
\$39.20 int. for 1 yr. 7 mo. 18 da.

The aliquot parts method is not easily applicable when accurate interest is required.

The Formula Method. The chief objection urged against the formula method is that the principles underlying the calculation of interest are so elementary that there is no occasion to substitute mechanical rules for the application of a few simple principles. If the pupil knows the general formula, $I = PRT$ and can solve a simple equation he can use the formula to advantage. If the habit of employing equations in the solution of problems has been formed, such formulas as the above may be used to advantage.

The Cancellation Method. The method is easily taught and is a favorite with many teachers. The following illustration will show the use of the method.

Find the interest on \$480 for 3 mo. 18 da. at 7%.

Solution:
$$\frac{\overset{36}{\cancel{108}} \times 7 \times \overset{4}{\cancel{480}}}{\underset{3}{\cancel{360}} \times \underset{25}{\cancel{100}}} = \$10.08$$

Explanation: 3 mo. 18 da. = 108 days. Seven is used as a multiplier because the rate is 7%. One hundred is used in the denominator because 7% is $\frac{7}{100}$.

Interest Tables. Interest tables are now extensively used and it is well to acquaint pupils with the use of such tables. Pupils will also be interested to construct a portion of an interest table. Practice will enable one to find results

quickly by use of an interest table. A small portion of an interest table is given below.

Beginning at 2 months, rate, 4%.

Total Days	$1000	$2000	$3000	$4000	$5000	$6000	$7000	$8000	$9000
60..	6.66	13.33	19.99	26.65	33.33	39.99	46.66	52.22	59.99
61..	6.77	13.55	20.33	27.10	33.89	40.66	47.43	54.21	60.99
62..	6.88	13.77	20.66	27.55	34.44	41.32	48.21	55.10	61.99

The table shows that the interest on $3,000 for 60 days at 4% is $19.99. The interest on $300 for 60 days is $1.999 and on $30 it is $.1999. The interest on $7,000 for 62 days at 4% is $48.21. By moving the decimal point the interest can easily be found on $700, $70, $7, or $0.70 at 4%.

By using the table we may find the interest on $5,486 for 61 days at 4% as follows:

The interest on $5,000 for 61 days at 4% is $33.89.
The interest on $400 for 61 days at 4% is $2.71.
The interest on $80 for 61 days at 4% is $0.5421.
The interest on $6 for 61 days at 4% is $0.0406.
Therefore, the interest on $5,486 for 61 days at 4% is $37.1827, or $37.18.

Similar tables may be constructed for any number of days and at any desired per cent. Such tables are much used by banks, insurance offices, and trust companies. The tables are arranged in convenient form and they greatly lessen the labor of computing interest. Several different arrangements of interest tables are published.

Indirect Cases. Indirect cases in interest have a limited application, but it is a mistake to assert that all indirect cases should be omitted. Questions regarding the rate of interest, or income, which a certain investment will yield

are of importance. For example, a man may wish to know what rate of interest he will receive on an investment of $80 in stock which pays an annual dividend of $5. The question evidently is: What per cent of $80 is $5? This is one of the type problems of percentage. Unless otherwise stated, the time in interest problems is assumed to be one year.

Problems in which the principal is required also occur in practical situations. However, such a problem as the following is unreal and impractical. "A man wishes to lend a sum at 6% such that the interest in 2 yr. 4 mo. will be $105. How much must he lend?" Teachers should not hesitate to omit types of problems that do not occur out of the classroom.

Problems in Simple Interest. Many of the problems in simple interest should be solved orally, and the pupils should be encouraged to use the shortest method to secure the desired result. In general, however, it is better to postpone the consideration of special short methods until the pupils are reasonably familiar with the usual methods.

The six per cent method, previously considered, may be abridged, and there is no reason why the short form should not be taught after the pupil is familiar with the more expanded form.

If required to find the simple interest on $420 for 5 mo. 18 da. at 6%, we may proceed as follows:

$$\frac{\overset{28}{\cancel{168}}\times\$420}{\cancel{6}\times 1000}=\$11.76$$

The 168 is derived from the reduction of 5 mo. 18 da. to days. We divide by 6 because we take $\frac{1}{6}$ of the number of days. We divide by 1000 because, having taken $\frac{1}{6}$ of the

number of days, we must use the result as mills, and mills stands in thousandths place.

What is the simple interest on \$480 at 7% for 5 mo. 12 da.

$$\frac{162\times7\times\$480}{6\times6\times1000}=\$15.12$$

We divide by the first six because we must take $\frac{1}{6}$ of the 162 days; by the second 6 in order to find the interest at 1%; we multiply this result by 7 in order to obtain the interest at 7%.

One more problem will illustrate sufficiently this abridged form. Find the simple interest on \$840.24 for 3 mo. 24 da. at 5%.

3 mo. 24 da. = 114 da.

$$\frac{114\times5\times\$840.24}{6\times6\times1000}=\$13.30$$

This method often offers opportunity for cancellation.

Computing the Interval between Dates. In computing the time between two given dates, the teacher should require the pupils to use the method prevalent in the community. In some localities the time is computed in days when the interval is less than a year, and in years, months, and days when the interval is greater than a year. In other localities the time for periods greater than a year is found in years and days. The time from April 14, 1926, to November 8, 1928, may be computed by any of the following methods:

(*a*)

1928	11	8
1926	4	14
2	6	24

The result is 2 yr. 6 mo. 24 da.

(*b*) The time from April 14, 1926, to April 14, 1928, is 2 years. From April 14 to October 14 is 6 months. From October 14 to November 8, is 25 days. The result is 2 yr. 6 mo. 25 days.

(*c*) From April 14, 1926, to April 14, 1928, is 2 years. From April 14 to November 8 (counting the exact number of days in each month) is 208 da., or 6 mo. 28 da. The entire time is therefore, 2 yr. 6 mo. 28 da.

There are sections of the United States in which each of the three methods is recognized as correct. In some states both the day a note is given and the day it matures are considered in computing the time.

Some texts contain a table showing the number of days from any day of one month to the same day of any other month within a year. Such a table is found on page 300.

CHAPTER XXII

COMPOUND INTEREST

Compound interest is the interest that accrues by making the interest due at the close of any interval, for which the interest is made payable, a part of the interest bearing debt for the next succeeding interval. Interest may be compounded annually, semi-annually, quarterly, or at any other interval, according to agreement.

Simple and Compound Interest Compared. At compound interest the amount of $100 for 10 years at 6% would be $179.08, while at simple interest the amount is $160. For example, find the interest on $500 for 3 years, 3 months, 15 days at 4%, interest compounded annually.

Solution: $500 = principal for first year

.04

$ 20 interest for first year

$500

$520 amt. for 1st year; principal for 2d yr.

.04

$ 20.80 interest for second year

$520

$540.80 amt. for 2d yr.; principal for 3d yr.

.04

$ 21.6320 interest for 3d year

$540.80

$562.432 amount for 3d yr.; principal for 4th yr.

The interest on $562.432 for 3 mo. 15 da. at 4% is $6.56.

$562.43 + $6.56 = $568.99 = amt. for 3 yr. 3 mo. 15 da.

$568.99 − $500 = $ 68.99 comp. int. for 3 yr. 3 mo. 15 da.

When compound interest is payable semi-annually or quarterly, we find the amount of the given principal for the first interval, and make it the principal for the second interval, etc. When the principal contains years, months, and days, as in the problem on page 291, we find the amount for the nearest exact interval and upon this amount compute the interest for the remaining months and days. This is then added to the last amount before subtracting the original principal.

The chief use of compound interest is among savings banks, building and loan associations, and insurance companies. Tables are generally used.

The following table shows the compound amount of $1 at various rates for intervals of 1 to 25 years:

COMPOUND INTEREST TABLE

Amounts of $1.00 compounded at various rates

Periods	1%	1½%	2%	2½%	3%	3½%	4%	4½%	5%	6%	Periods
1	0.0100	1.0050	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1
2	1.0201	1.0302	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	2
3	1.0303	1.0457	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	3
4	1.0406	1.0614	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	4
5	1.0510	1.0773	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	5
6	1.0615	1.0934	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	6
7	1.0721	1.1098	1.1487	1.1887	1.2299	1.2723	1.3169	1.3609	1.4071	1.5036	7
8	1.0829	1.1265	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	8
9	1.0937	1.1434	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	9
10	1.1046	1.1605	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	10
11	1.1157	1.1779	1.2438	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	11
12	1.1268	1.1956	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	12
13	1.1381	1.2136	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	13
14	1.1495	1.2318	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	14
15	1.1610	1.2502	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	15
16	1.1726	1.2690	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	16
17	1.1843	1.2880	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	17
18	1.1961	1.3073	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	18
19	1.2081	1.3270	1.4568	1.5977	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	19
20	1.2202	1.3469	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	20
21	1.2324	1.3671	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	21
22	1.2447	1.3876	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	22
23	1.2572	1.4084	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8198	23
24	1.2697	1.4295	1.6084	1.8087	1.0328	2.2833	2.5633	2.8760	3.2251	4.0489	24
25	1.2824	1.4509	1.6406	1.8539	1.0938	2.3632	2.6658	3.0054	3.3864	4.2919	25

The compound amount of \$300 at 4% for 5 years, interest compounded annually is $1.2167 \times \$300 = \$365.01$. The compound amount of \$450 for 6 years at $3\frac{1}{2}\%$ is $1.2293 \times \$450 = \$553.185$.

The compound amount of any given sum at a given rate payable semi-annually, equals the compound amount of the same sum for twice the time at one-half the rate payable annually. The compound amount of any sum at a given rate payable quarterly equals the compound amount of the same sum for four times the time, at one-fourth the rate, payable annually. For example, 3 yr. 6 mo. at 4% semi-annually will yield the same compound amount as 2% for 7 years, the interest being compounded annually. The compound amount of a given sum for 3 years at 8% compounded quarterly is the same as for 12 years at 2% compounded annually.

Applications. The subject of compound interest affords the teacher an excellent opportunity to direct attention to the importance of savings banks. If the state laws governing savings banks are rigid enough to make them desirable institutions, the teacher should direct attention to the fact that such institutions receive small as well as large deposits, and pay compound interest on all deposits left for a certain length of time. An appeal to economy and thrift may be made by calling attention to the rapidity with which even small deposits increase, if they are regularly made and left in the bank at compound interest for a few years.

Ask the pupils to find out how much a boy would have in a savings bank at the end of 5 or 10 years if he deposited \$1 each week and made no withdrawals, the bank paying 3% interest, compounded quarterly. Most pupils will be surprised at the rapidity with which even small savings accumulate.

The following illustration is taken from *A Scrap Book of Elementary Mathematics*, by William F. White (p. 47). It shows the enormous results obtained when compound interest is computed for long periods. The illustration will impress this fact upon the minds of pupils.

"At 3% one dollar put at interest at the beginning of the Christian era to be compounded annually would have amounted in 1906 to $(\$1.03)^{1906}$, which in round numbers, is \$3,000,000,000,000,000,000,000,000. The amount of \$1 for the same time and rate, but at simple interest, would be only \$58.18."

SUGGESTED READINGS

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CHAPTER XXIII

BANKING: CORPORATIONS: BUSINESS PRACTICE

It is not the duty of the school to acquaint pupils with all of the technicalities involved in banking, but pupils should know the functions of a bank in modern commercial life; the principal duties of the bookkeepers and the cashiers. Deposit slips and blank checks should be obtained from local banks, and children should be taught how to make deposits and draw and endorse checks. Attention should be directed to the fact that the figures and words stating the amount of a check should be written so that they cannot be changed easily. The endorsement should be written on the back of the left end of the check.

The chief functions of a bank are:

1. To receive money on deposit and to pay checks drawn on deposits.
2. To lend money on promissory notes.
3. To sell drafts on other banks and collect drafts on persons.
4. To issue its own promissory notes, which serve as currency.

In the United States only national banks can issue their own notes. Show the pupils some bank notes issued by a national bank or by a federal reserve bank.

The School Bank. Problems involving banking should be made real to the pupils. The nearer the approach to actual business the greater will be the interest in the work. The school may have a bank in which certain pupils act as cashier, paying teller, receiving teller, and bookkeeper. Inexpensive paper money, deposit slips, and checks may

be secured. More than ninety per cent of the business of the country is carried on by checks and drafts, but few pupils have seen either. In most localities a local banker will be glad to talk to the pupils on topics of interest related to banking. In many progressive schools, school savings banks are now maintained.

Systematic thrift is encouraged, and the pupils acquire a great deal of practical knowledge in regard to the operation of such a bank.

Negotiable Paper. Pupils should become familiar with the principal forms of promissory notes and checks. These instruments are an important factor in business transactions, passing from hand to hand as a substitute for money.

No exact form need be followed in order to make a contract a negotiable instrument, but custom has prescribed forms that are very generally used. A negotiable instrument, whether it is a promissory note, bill of exchange, or check, must be:

(*a*) In writing. (No oral contract is negotiable.)

(*b*) Properly signed. (The name is not necessary; any mark intended to serve as a signature will suffice.)

(*c*) Negotiable in form. (If made payable to a particular person or persons only, it is not negotiable.)

(*d*) Payable in money only. (The value is then definite and certain.)

(*e*) The amount must be definitely stated.

(*f*) It must be payable absolutely. (Not upon certain conditions.)

(*g*) To the order of a designated payee or bearer. (To a person or persons who can be identified when the note matures.)

(*h*) On demand or at a certain time. (Not at a time contingent upon some other event, but at a time sure to come.)

Promissory Notes. The party who makes the note and whose promise is stated is called the *maker*. The party to whom the promise is made is called the *payee*. If the note is without interest, this fact may be stated in the note or the words "with interest" may be omitted.

The payee of a negotiable note may transfer his rights to another by writing his name on the back of the note. This is called *endorsement*, and the person to whom the note is transferred is called the *endorsee*.

$1800.00 Chicago, Illinois
September 28, 1927

Sixty days after date *I* promise to pay to the order of *James Riley*,

Eighteen Hundred Dollars *$1800.00*

without interest.

Value received. *John Baker*

Figure 3

Ask the pupil who is the maker of the above note? Who is the payee? When is the note due? Who pays the note when it matures? To whom? How much must be paid when the note matures? Who gets the check after it is paid? Why are the words "value received" put in a note? Is the above note negotiable? Suppose the payee wishes to sell the note before it matures, how should he endorse it?

If the note is endorsed by the words "pay to William Jones" it cannot be transferred again. This is called a "restricted endorsement." The holder of the note may simply write his name across the back of it, as shown in

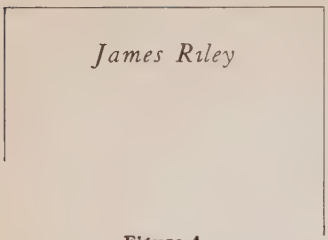

Figure 4

Figure 4. This is called an endorsement in blank. It is a very common form of endorsement and makes the note payable to bearer. The question of liability incurred by the endorser of a note should be discussed. The extent to which the endorser of a note limits his liability by using the words "without recourse" should be considered.

There are numerous kinds of notes and drafts involving more or less technicalities. The teacher should acquaint the pupils with only the simpler forms in common use. In treating the subject of negotiable paper a good business arithmetic will be of value to the teacher.

$1800.00 Chicago, Illinois
September 28, 1927

On demand *I* promise to pay to the order of *James Riley*,

Eighteen Hundred Dollars *$1800.00*

without interest.

Value received. *John Baker*

Figure 5

When money is borrowed from a bank the security given may be (*a*) real estate, (*b*) collateral, (*c*) personal. These should be discussed in the class and given practical setting by use in the school bank. Many pupils have the idea that successful men and women do not need to borrow money. Almost all successful persons and firms borrow from banks.

It is often good business policy for a manufacturer, for example, to borrow to pay for raw materials and repay the loan from the receipts for the finished product. Most loans made by commercial banks are short time loans; usually for 30, 60, or 90 days. Many "demand" loans are made. Savings banks usually make loans for longer periods.

Discounting Notes. Suppose that James Riley, the payee of the note on page 297, sells the note to a bank on October 23. It is evident that the bank could not afford to pay $1,800 for it because that amount will not be paid to the holder of the note until it is due — November 27; the bank would lose the interest on $1,800 from October 23 to November 27. The interval between the date a note is discounted and the date of maturity is called "the term of discount." The difference between the value of a note at maturity and the amount paid for the note when it is sold is called *bank discount.* It is evident that bank discount presents no new mathematical principles. It is a method of collecting interest in advance, and is commonly used by commercial banks.

A note that does not bear interest matures at its face value. If it is interest-bearing from date, the interest which will have been earned at the date of maturity must be added to the face of the note to determine its maturity value.

The first step in calculating discount on a note is to find the maturity value, since the purchaser must know this before he can determine how much he can afford to pay for it.

In computing the term of discount banks usually count the exact number of days. If a note is due a certain number of months after date, it will mature on the same day of the month as the date of the note.

For convenience the following table is often used by bankers to find the days intervening between dates.

TABLE X

From any day	To the same day of the next: January	February	March	April	May	June	July	August	September	October	November	December
January	365	31	59	90	120	151	181	212	243	273	304	334
February	33	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	122	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

The exact number of days from any day of one month to the same day of another month, within a year, is found by starting at the name of the first month in the left-hand column and following across to the column of the last named month. Thus, to find the number of days from April 18 to August 24, we note that it is 122 days from April 18 to August 18, and then we add 6 days (August 18 to August 24) which gives 128 days as the exact time.

Suppose we wish to find the bank discount on the note given on page 297, if the note is discounted on October 23 at 6%.

$1,800 = maturity value of the note.

Time from October 22 to November 27 is 36 days; this is the term of discount.

Interest on $1,800 for 36 days at 6% is $10.80; this is the bank discount.

The seller of the note would receive \$1,800 − \$10.80, which is \$1,789.20.

If the note had been drawn with interest at 6% the maturity value of the note would have been \$1,800 plus the interest on \$1,800 at 6% for the time of the note (60 days). This would have been the basis for computing the bank discount.

Partial Payments. The subject of partial payments is no longer of great value because of changes that have been made in the forms of notes. The modern bank is rendering the subject relatively unimportant. The numerous state rules should not be taught. It is best to teach the United States Rule, if any, and to solve only a few problems.

CHAPTER XXIV

METHODS OF PAYING BILLS AT A DISTANCE

Some attention should be given to the common methods in general use for paying bills at a distance. The advantage and disadvantages of each method should be briefly discussed.

Debts between persons in the same community are usually discharged by means of actual cash or by checks. However, actual cash is seldom sent to a distant point to discharge a debt. The expense and the chance of loss are too great to justify this method. Small bills and coins are sometimes sent by letter, but this practice is not common, and the majority of remittances are made by one of the following methods: Personal check, draft, postal or express money order.

The Personal Check. The personal check is a convenient method to use in discharging debts at a distance. Its use for this purpose has increased rapidly in recent years. However, the use of the personal check for such purposes has certain disadvantages. The person or firm receiving the check may not know whether the check is good and it may take several days after the check has been deposited at a local bank before this can be determined. Banks receive such checks "for collection." In some communities a small charge called exchange is made by the bank for collecting "out of town" checks.

The Bank Draft. Large accounts are commonly paid by drafts. For example, Mr. Andrews of Minneapolis wishes to pay a bill in New York. He buys a New York draft at his bank in Minneapolis. The Minneapolis bank

has a checking account in some New York bank and a check in favor of Mr. Andrews' creditor, for the desired amount, is drawn on this account by the Minneapolis bank and this check is sold to Mr. Andrews, who sends it to his creditor.

All banks have checking accounts with banks in the great financial centers of the country. The extensive use of bank drafts is made possible by the development of our modern banking system. Thousands of bank drafts are sent each business day and these drafts are balanced against one another in great financial centers. This practice greatly reduces the amount of actual cash which it would otherwise be necessary to send.

Postal and Express Money Orders. The payment of small accounts is frequently made by postal or express money orders. The fees for such orders are listed in most arithmetics and they may be learned at any postoffice or express office. The fees for postal and express money orders are not the same for all amounts.

Postal money orders are not issued for more than $100 and express money orders are not issued for more than $50. One may, however, purchase any number of such orders. Large accounts are not settled by money orders, because the cost is high and several separate orders would need to be sent.

Accounts. All pupils should be taught how to keep simple accounts. The average man or woman finds but little need for an elaborate form of accounts, but boys and girls, farmers, housewives, in fact any one who uses money should keep simple accounts in order to economize resources and spend wisely. Such accounts as are here advocated can be intelligently kept by boys and girls in the elementary grades. The best of the recent arithmetics contain simple systems of accounts and the topic will not need to be discussed here in further detail.

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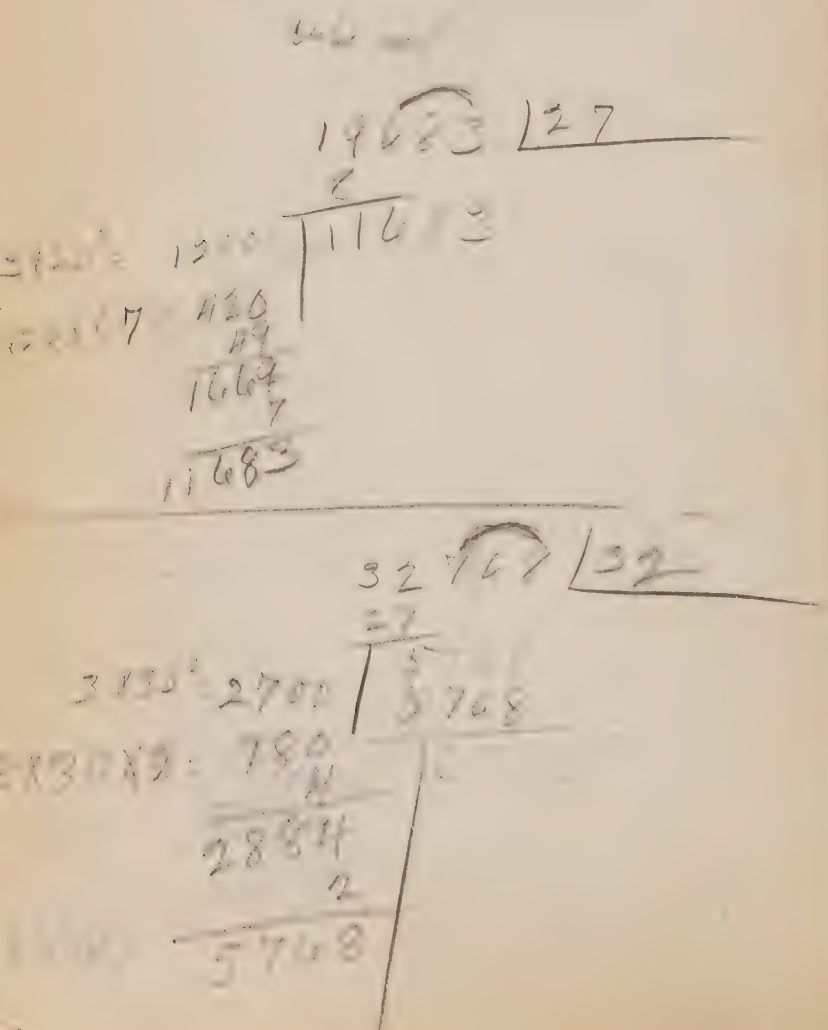

CHAPTER XXV

COMMISSION

Many workers receive an agreed amount for their services. This amount whether quoted in terms of the hour, the day, the week, the month, or the year, is usually called wage or salary. Thus, a certain shop employee may receive wages of $5.60 a day; a clerk $24 a week; a teacher $120 a month, and a bank president $12,000 a year. In many cases, however, a worker is paid a certain per cent of the amount of business done. Thus, a real estate agent, who sells a house for $6,000 and receives 2% of the amount of the sale, is paid $120. A boy who takes subscriptions for a magazine, and who is paid 20% of the subscription price, will receive 40 cents on each $2 subscription. Compensations which are fixed as a certain per cent of the amount of business done are called *commissions.*

The importance of the commission business in this country is largely the result of our industrial and commercial development. The rural districts send enormous quantities of grain, hay, and cotton and butter, eggs, meat, and other food stuffs to the cities. Most of the transactions involved in the sale of these products are through *commission merchants.* The commission merchant sells to wholesalers, to retailers, or others. He receives a certain per cent of the selling price as his *commission.* The remainder of the selling price is sent to the party who made the shipment. Insurance agents usually receive as their compensation a certain per cent of one or more of the annual premiums. Real estate agents are usually paid a certain per cent of the amount involved in the transaction.

Most agents who solicit sales for books and magazines are working on commission. Many men and women who travel as salesmen receive a commission on the total amount of their sales. Some receive a stated salary, and in addition a commission on all sales over a certain amount. Many clerks receive a stated salary and a commission, which is a certain per cent of their sales above a specified amount.

It should be noted that the agent who works on commission does not own the property which he sells. He is acting for the owner.

Pupils will find it interesting and valuable to list the various kinds of work in their communities which are on a commission basis. The pupil should see that economic conditions demand that there shall be commission agents. The farmer who ships his cotton or his wheat to the central market does not usually go the market.

Making Commission Interesting and Practical. The subject of commission may be made interesting to both city and country pupils, if some of the introductory problems are based on local conditions, especially on those involving the sending of farm products to the city. Pupils should get an insight into the actual operations of commission. Many of the pupils will, later in life, either receive some compensation as commission or have business relations with those who do. The problems given should involve the direct operations only, and should be in harmony with actual business practices. Such a problem as the following was formerly common in textbooks. A man sends his agent \$4,830 to invest; after deducting his commission of 5%, how much does the agent invest? Such problems do not represent actual business conditions and should not be used. No one sends an agent a stated amount from which he is first to deduct his commission and then invest the remainder.

Technical Terms of Commission. Such terms as agent,

principal, consignment, consignor, consignee, remittance, net proceeds, account sales, and account purchases, are technical terms of the subject. These terms are defined in the textbooks and need no comment here. The language of commission causes the pupil much more difficulty than the mathematics of the subject.

Illustrative Problems

1. A man works in a store for \$15 a week and a commission of 2% on all his sales. Last week his sales were \$655. How much did he earn?

2. During the summer vacation Florence sold books on a commission of 20%. Her sales were \$1,460. How much was her commission?

3. A fruit grower shipped 360 boxes of apples to a commission merchant, who sold the apples at \$1.40 a box. The freight, drayage, and storage amounted to \$165.45. The commission was 3%. How much did the grower get?

4. A man was offered a salary of \$2,800 a year, or 3% on all sales. He chose the first plan and sold \$98,000 worth of goods during the year. How much would he have gained or lost by the second plan, if his sales had been the same?

CHAPTER XXVI

STOCKS AND BONDS

The subject of stocks and bonds is generally considered to be one of the most difficult applications of percentage. The difficulty is due, however, not to the mathematics involved, but to the fact that the language of the subject is frequently unfamiliar to the teacher, and usually unfamiliar to the pupil.

The enormous amount invested by the people of the United States in stocks and bonds makes this subject one of practical interest to millions of people. It is estimated that about $\frac{1}{5}$ of the total property of the United States is in the form of stocks and bonds.

Corporation Stocks. The best way to approach the subject in order to make it real to the pupils is to let the pupils organize a corporation. When such an organization is completed the pupil will have a better knowledge of the meaning of the technical terms of stocks and bonds than he would have had from a study of the terms in a more formal manner. Corporations are assuming control of many business enterprises and are replacing the individual in the larger business ventures. Partnerships were very common in the early days in this country, but the enormous amounts necessary to finance many modern business projects have made it impossible for a few individuals to supply the necessary capital. Some of our large corporations, such as the railroads, have thousands of stockholders.

Before a corporation can be organized it must secure a commission from the Secretary of State. The persons

commissioned by the Secretary of State are permitted to take subscriptions for the stock. After the legal amount of stock is subscribed a meeting is held for the purpose of effecting a temporary organization. A constitution is adopted and the officers provided for in the constitution are elected. The name and purpose of the organization, its location, the number of shares and par values of each share, the officers and the time for holding regular meetings of the stockholders are determined upon and a statement of these facts is filed with the Secretary of State. A *charter* is then issued and the organization is completed. The charter defines the duties, powers, and obligations of the corporation. The stock previously subscribed for is sold. The directors are empowered to buy, sell, contract debts, etc.

Suppose that the pupils organize a company with a capital stock of $100,000, the par value of each share being $100. There will then be 1000 shares of capital stock. Of course, the *par value* may be $500, or $50, or $10, a share, or any other convenient amount. Attention should be directed to the fact that the par value of a stock is not an indication of its market value. A stock having a par value of $100 a share may have a market value of only $30 or $40, or even less, or it may have a market value much above $100. The pupil who buys 20 shares of the stock will pay $2,000 for it, if the stock sells at its par value. For each share that one owns he is entitled to one vote at all meetings of the stockholders. When one pays for his stock he receives a "certificate of stock," stating the number of shares bought, the par value of each share, etc. The owner of stock is entitled to participate in the profits in proportion to the number of shares owned.

A certificate of stock is not a promise to pay. It is evidence that the owner of the certificate holds an interest in the corporation. No interest return is guaranteed and

the stock certificate has no maturity date. The earnings of a share of stock depend upon the profits of the company.

Corporations often issue both *preferred* and *common stock*. A preferred stock certificate provides that dividends up to a certain limit, usually 5% or 7%, are to be paid before any dividends are paid on common stock. Holders of preferred stock have first chance at dividends, but their dividends are limited and nothing is guaranteed.

Suppose that at the end of the first year the corporation organized by the pupils has made a profit of $12,000 above all expenses. The directors may decide to declare this as a dividend, or all or a part of it may be put into a "reserve fund." If it is declared as a dividend and if the holders of preferred stock are entitled to 6%, this must be paid before any dividend is paid to the holders of common stock. Let us suppose that $30,000 of the entire stock ($100,000) is preferred stock. The holders of this stock are entitled to 6% of $30,000, or $1,800. A dividend of $6 is paid on each share. The holders of the common stock are entitled to the balance of the earnings, or $10,200. This divident will be divided among the holders of $70,000 worth of stock. Each holder of common stock will, therefore, receive $14\frac{4}{7}$% dividend on each share. Since this is much higher than the current rate of interest, the common stock probably will sell above par, that is, at a premium.

Suppose that at the end of the second year the corporation has made a profit of but $1,800. If dividends are declared the holders of preferred stock are entitled to this entire amount, since it is just 6% of the face value of the preferred stock. The holders of common stock would receive no dividends and their stock would likely sell below par. In case the net earnings of the corporation are not sufficient to pay 6% dividends on the preferred stock, the earnings of the corporation, if declared as dividends, will be divided pro rata among the holders of the preferred stock.

The teacher should emphasize the fact that, if a stock pays a dividend of 7%, this means that 7% of the par value of each share is paid to the owner. The market value of the stock may be above or below par, but dividends are always based on the par value.

Bonds. Corporations may borrow money, pledging all or a part of their property as security in the same way as individuals. When a large amount of money is to be raised, it is usually done by issuing bonds. A bond is a mortgage note, upon which the corporation agrees to pay interest, and to the payment of which the entire property and business of the corporation is pledged. Unlike a stock certificate, a bond draws a guaranteed per cent of interest, and it has a maturity date. The owner of stock will receive whatever per cent of the par value is declared as a dividend, but the owner of a bond will receive no more or less than the amount provided in the bond.

Bonds may be issued by national and state governments, counties, townships, cities, school districts, or corporations. These are declared by law to be "bodies corporate."

Bonds made payable to the owner or his assignee are called registered bonds; bonds made payable to bearer are termed coupon bonds. In coupon bonds the interest is provided for in attached notes or "coupons." In the case of registered bonds the interest is sent directly to the owner. The holder of a coupon bond must surrender a coupon when each interest payment is made.

If a bond pays more than the current rate of interest and is safe, it usually sells at a premium. Some bonds, such as the bonds of the United States Government, that pay less than the current rate of interest may sell above par because of the great security and because they are tax exempt.

The prices of many important stocks and bonds are quoted in the daily papers. Stocks and bonds are not often

sold at par. Market values often fluctuate widely and an intimate knowledge of corporations and market conditions is necessary if one expects to invest wisely. A *broker* is assumed to know the stock and bond market and to be able to advise his clients when to buy or to sell stocks and bonds. The fee paid him for his services is called commission or brokerage. It should be remembered that brokerage is always computed as a certain rate on the par value whether stocks and bonds are bought or sold. A knowledge of the technicalities of the broker's office is not of value to most citizens, and need not be taught in the schools.

Rate of Income from Investments. It is very important that an investor should know what rate of income on the investment is yielded by a stock or bond bought at a given price. This computation is very simple and involves one of the three type problems of percentage. For example, if I buy a 6% stock at \$120, including brokerage, I receive \$6 in dividends on each share, but in order to receive the \$6 I invested \$120. The rate of income that I actually receive on my investment is whatever per cent \$6 is of \$120. Therefore, I receive 5% on my investment. Similarly, 4% stock bought at \$80 yields 5% on the investment.

Illustrative Problems

A man sells 160 shares of stock at 90; brokerage 15 cents per share. How much does he receive?

1. \$90 − \$.15 = proceeds from each share.

160 × \$89.85 = \$14,376 = proceeds from 160 shares.

2. What sum must be invested in 5% bonds at 110, without brokerage, to yield an annual income of \$450?

Each bond yields \$5 dividend annually (5% of par value).

To yield \$450 the number of bonds must be \$450 ÷ \$5, which is 90.

Since each bond cost \$110, the entire cost was $90 \times \$110 = \$9{,}900$.

3. What rate of income does an investor make on stock that pays a dividend of 5% if he buys it at 80, including brokerage?

\$5 = income on each share.

\$80 = cost of one share.

The return on the investment is whatever per cent \$5 (the dividend) is of \$80 (the amount invested). \$5 is $6\frac{1}{4}\%$ of \$80.

4. What is the market price of 7% stock that yields an income of 10% on the investment, no brokerage?

\$7 = income on each share.

\$7 = 10% of the amount invested in one share, therefore, the amount invested is \$70.

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CHAPTER XXVII

TAXES AND REVENUE

The subject of taxes is one in which most of the homes represented by the pupils are directly interested. Once or twice each year the payment of taxes becomes a very real matter and the entire subject can be made interesting to the pupils, if most of the problems are based on local conditions. The subject of taxes offers an excellent opportunity to emphasize certain duties and responsibilities of citizenship and to correlate arithmetic closely with practical life. The social and economic phases of the subject should receive special emphasis.

Why Taxes Are Necessary. The first essential is that the pupils shall understand why the levying and collecting of taxes is necessary. It should be explained that a tax is a money levy made by a government — national, state, county, township, or city — to defray all or a part of its expenses. Pupils should be asked to name several items of expense of the nation, state, county, township, and city; such as the salaries of state officials, the erection and maintenance of public hospitals, asylums, prisons, state schools, and public buildings. The county must raise money to defray expenses, such as those incurred in paying the salaries of county officials, the erection and maintenance of the courthouse and other county buildings, the building and repairing of roads and bridges. The city must levy a tax to maintain the public schools, to provide proper sewerage and other sanitary conditions, to provide police and fire protection, to pay the salaries of city officials, to maintain public playgrounds and parks.

Ask the pupils to compare the taxes which a man pays each year with what he would have to pay to secure equivalent service and protection, if no taxes were levied. If taxes were not levied he would have to pay for the education of his own children, and this item alone would, in most cases, be larger than his annual taxes. He would also need to provide for protection from fire and theft. The pupils should see that a just levy of taxes is not only wise but economical for the individual citizen.

Taxes Are Increasing. As the number of things for which the government—whether national, state, county, township, or city—must spend money increases, it is obvious that the amount to be raised by taxation must increase. The necessities, conveniences, and comforts which are provided must be paid for, and most of the money to make the payments must be raised by taxation. Ask the pupils to make lists of the purposes for which taxes are expended by their state, county, township, and city, and to consider the probable expenditures which were made for these same purposes fifty or a hundred years ago.

Kinds of Taxes. For purposes of taxation there are two general kinds of property tax — real estate and personal. Real estate comprises property that is not easily moved, such as land, mines, quarries, buildings, railroads, and forests. Personal property is easily moved, such as money, stocks, bonds, live stock, and household goods. Separate assessments are made for each kind of property, and in some localities the taxes on personal property and on real estate are due on different dates. Pupils should be asked to explain why the rates of city and county taxes in a given locality differ. What protection or conveniences does the man who resides in the city have that the man in the country does not have? Is it reasonable that the resident of the city should be assessed for this protection and convenience?

Besides the property tax just mentioned, there are several other kinds of taxes. In some states a poll tax of a specified sum is levied on all male citizens over twenty-one years of age. Franchises which are granted by municipalities or counties are often subject to special taxes. In many localities a special dog tax is levied. A special tax is also levied in some states on all inheritances over a certain prescribed minimum. Corporations pay a special tax on their profits, and an income tax is levied on persons having an income above a certain amount. In certain states other types of taxes are now in force. Among these taxes the following may be enumerated: Luxury taxes, amusement taxes, occupation taxes, and severance taxes. The latter is really a license tax imposed on units of production of natural wealth such as oil, ores, and timber.

Levying and Collecting Taxes. The method of levying and collecting taxes varies in the different states and a detailed discussion of the various methods cannot be given here, but the teacher should not fail to ascertain the method used in the state and in the county in which he is teaching, and to familiarize the pupil with it. Such a study should acquaint the pupil with the meaning and use of the following terms: Valuation, assessor, assessment, board of review, board of equalization, tax collector, and delinquent taxes.

How Tax Rates Are Determined. The method by which the rate for a given kind of tax is determined should be clearly understood. The state legislature determines the amount to be spent by the state. The assessed value of the property of the state is ascertained and from these two items the rate of the state tax is determined. The total amount to be raised is divided by the total assessed valuation. For example, if a tax of $63,400 is to be raised and the assessed valuation is $3,760,450, the tax rate will be $.01686.

Tax rates are usually expressed as so many mills on each dollar of valuation or as so many dollars on each hundred or on each thousand dollars of assessed valuation, or as a per cent. In the example just given, the state tax would be 17 mills on each dollar.

The pupil should be told what body is authorized to determine the amount of each of the above taxes. In many localities it is the custom to assess property at $\frac{1}{2}$ or $\frac{2}{3}$ of its real value.

The time and labor of computing taxes are reduced by the use of tables from which the tax on various sums at specified rates may be easily found.

Base Problems on Local Conditions. Most teachers can obtain assessment blanks and several tax receipts for class use. Problems based on local levies and assessments are of much more interest to the pupil than most of the problems of the textbook. Details in regard to state revenue laws may be secured by addressing the Secretary of State.

Duties and Revenues. The subject of national duties and revenues is closely related to that of taxes, and may be presented in a similar manner. Among the chief items of national expense may be mentioned the maintenance of the Army and the Navy, Diplomatic and Consular Service, Indians, Internal Improvements, Interest on the Public Debt, Erection and Maintenance of Public Buildings, Salaries of Government Officials, and Pensions. The expenses of the National Government are defrayed by the receipts from duties and custom, internal revenues, sale of public lands, income and corporation taxes, etc.

The study of revenues introduces the question of *specific* and *ad valorem* duties, the tariff, and kindred subjects. The teacher must use his discretion as to how fully these subjects should be considered in a class. The mathematics involved in the problems of taxes and national revenues is so simple that illustrative problems are not necessary

here. The entire subject should acquaint the pupil with local, county, state, and national methods of defraying necessary expenses. The mathematics involved should be considered as subordinate to the general informational and social phases of the subject.

CHAPTER XXVIII

INSURANCE

Insurance is such an important factor in modern life that the broader aspects of the subject should be taught in the schools. The attempt should not be made to explain the technicalities involved in the various kinds of policies. Only the more common types of property and personal insurance should be considered. The pupils should know the principal kinds of property insurance, such as fire, theft, crop, tornado, hail, and marine, also of personal insurance, including life, health, and accident.

Kinds of Insurance. Insurance involves a contract guaranteeing, in consideration of a payment called a *premium*, an indemnity in case of loss resulting from certain causes. When such a contract is entered into by a number of persons who agree to share mutually the losses, it is known as mutual insurance.

When a company is organized as an investment, and the stockholders agree to share the profits and the losses, it is known as a stock company. Sometimes the principles of the mutual and the stock company are combined into a mixed company. In such an organization all of the earnings of the company above a specified dividend on the stock are divided. The company assuming an insurance risk is called the *insurer* or the *underwriter*.

The contract between the insurer and the insured is called a *policy*. It states the conditions under which the guarantee against loss is made, the time the policy is to be in force, the rate of premium, and other necessary facts.

The principal kinds of property insurance are: Fire,

tornado, lightning, burglary, automobile, live stock, crop, marine, plate glass, steam boilers, and transit. The principal kinds of personal insurance are life, health, and accident.

The *premium* is the amount paid for the insurance. In mutual companies the cost of insurance depends largely upon the losses suffered by members of the company. In stock companies a definite premium is charged for insuring for a given amount for a given time. In mutual companies the cost to the policy holder is called an *assessment.* The rate of insurance depends upon the nature of the risk, the type of policy and the period for which the policy is to be in operation. The rate is sometimes stated as a certain per cent of the face of the policy, but more frequently it is quoted as so much per $100.

Fire Insurance. Fire insurance on buildings is usually written for one, two, three, or five years. Stocks and merchandise are not often insured for more than a year, because of the variations in value. Insurance upon crops or grain may be for much shorter periods.

The rates of fire insurance vary with the construction of buildings, the purpose for which they are used, their location with reference to other buildings, the fire protection afforded by the community, etc. Thus, other things being equal, the rate on a brick building is less than on a similar frame structure. The rate on a frame livery barn is high because of the relatively great likelihood of fire in such a structure. The rate for a three year policy is usually two and a half times the rate for one year and the rate for five years is four times the rate for a year.

Fire insurance usually covers not only loss from fire, but also from smoke and water, and from damage in putting out a fire in adjoining buildings. It is customary to insure property for about 70% to 80% of its actual value.

Some policies contain an "average or co-insurance" clause. In such cases the company, will, in case of loss,

pay only such a part of the loss as the amount of the policy is of the amount of the insurance which the owner agreed to carry. Thus, when property worth $5,000 is insured under a policy which contains an 80% co-insurance clause, the owner agrees to carry insurance totalling 80% of $5,000 or $4,000. If the owner carries only $3,000 of insurance, and a fire causing a total loss occurs, the owner can collect only $\frac{3}{4}$ of the amount of his insurance. Insurance companies sometimes assume a risk and then reinsure a part of the risk in other companies.

Ask the pupils to find out the rates of insurance on various kinds of buildings in the community. Require them to account for the difference in rates. If a man constructs a building that is more nearly fireproof than other buildings of the same kind, does he thereby save on his insurance premiums? Should schoolhouses and courthouses be insured? Why? Is your schoolhouse insured? Why is a property usually not insured for its full value?

Consider the details of a local fire, and show what the duties of an insurance adjuster are. The teacher should secure some fire insurance policies and use them in the class. Some of the pupils may be acquainted with some one in the community who is agent for one or more insurance companies. The compensation paid to an insurance agent is commission. As far as possible the problems should be based on local conditions.

Life Insurance. The three general forms of life insurance policies are: *Straight life; limited payment life;* and *endowment.*

One who has a straight life policy pays the premium when due, during his entire lifetime, and the insurance company agrees to pay a certain sum to the beneficiary at the death of the insured.

The holder of a limited payment life policy pays the premiums for a fixed number of years, but the amount of

the policy is not payable until the death of the insured. Most limited payment policies are for ten, fifteen, or twenty years.

An endowment policy is one in which the company agrees to pay a certain sum at the end of a certain number of years, or to pay the amount at the death of the insured, if he dies before the expiration of the specified time.

The object and the principal advantages of each of these types of policies should be understood. If possible, the teacher should show the class a policy of each of these types, and the distinguishing features and relative advantages and disadvantages of each may be discussed.

A person may insure his own life or that of another. The necessity for a physical examination of the applicant for insurance and for the important facts of his family history should be made evident.

Premium Rates. The annual premiums differ greatly for different kinds of policies. The premium on the straight life policy is the lowest, since the payments continue until the death of the insured. The premium on the limited payment policy is higher and the shorter the time the payment of premiums is to continue, the higher is the premium. The premium on the endowment policy is relatively the highest.

The principal items which determine premium rates are mortality cost, operating costs, and reserve. An elementary treatment of insurance need not consider these items in detail. The premium charged for a given kind of insurance for a person of given age varies but little among the well established life insurance companies.

The following table of annual premiums per $1000, is taken from the rate book of one of the well-known companies. Some very interesting problems can be made from this table.

TABLE XI

Age	Straight Life	20-Payment Life	20-Year Endowment	Age	Straight Life	20-Payment Life	20-Year Endowment
21	19.62	29.84	48.63	41	34.16	43.82	54.60
22	20.06	30.31	48.77	42	35.39	44.90	55.20
23	20.51	30.80	48.96	43	36.70	46.04	55.85
24	20.99	31.31	49.14	44	38.08	47.25	56.56
25	21.49	31.83	49.33	45	39.55	48.52	57.34
26	22.01	32.37	49.53	46	41.12	49.87	58.20
27	22.56	32.94	49.73	47	42.79	51.31	59.14
28	23.14	33.52	49.95	48	44.47	52.83	60.17
29	23.74	34.13	50.18	49	46.46	54.45	61.31
30	24.38	34.76	50.43	50	48.48	56.17	62.25
31	25.05	35.42	50.69	51	50.62	58.01	63.01
32	25.75	36.11	50.96	52	52.91	60.07	64.41
33	26.50	36.82	51.26	53	55.35	62.06	67.05
34	27.28	37.56	51.57	54	57.95	64.29	68.84
35	28.11	38.34	51.91	55	60.72	66.69	70.81
36	28.98	39.15	52.28	56	63.68	69.26	72.97
37	29.90	40.00	52.67	57	66.84	72.01	75.31
38	30.80	40.89	53.10	58	70.22	74.98	77.91
39	31.91	41.81	53.56	59	73.83	78.16	80.73
40	33.01	42.79	54.06	60	77.69	81.60	83.82

This table indicates, for example, that if a person 25 years of age takes out a $1,000 Straight Life Policy, the annual premium will be $21.49. If he takes out a 20-Payment Life Policy for $1,000, the annual premium will be $31.83, and, if he takes out a 20-Year Endowment Policy for $1,000, the annual premium will be $49.33.

The textbook problems on insurance involve no mathematical difficulties, and if the pupil understands the general features of the subject he should have no difficulty in solving the problems.

CHAPTER XXIX

INVOLUTION AND EVOLUTION

Neither involution nor evolution is frequently used in business. A knowledge of squares and cubes, of square roots and cube roots is necessary in certain scientific work, but the scientist and the engineer use tables or the slide rule to secure the results desired. Babylonian tablets, which probably date back to about 2100 B.C., found in 1854, contained the squares of numbers from 1 to 60 and the cubes of numbers from 1 to 32.

Cube root has been eliminated from most courses in arithmetic. The study of cube root is being postponed until the pupil encounters the subject in algebra, or is omitted entirely. Square root is taught in most schools in the eighth grade. The chief reason for its retention in the course is the fact that it is needed in the mensuration of certain geometrical forms. For example, a knowledge of square root is necessary in finding the diagonal of a rectangle or of a cube, the altitude and area of an isosceles triangle, and the slant height of a pyramid or cone. If square root is applied to the solution of some real problems, pupils will be interested in it.

The Terms Used. The technical terms of involution and of evolution should be thoroughly understood. The meaning of digit, power, exponent, square and square root, and the radical sign should be familiar to the pupil. Time in computation will be saved if the pupil is required to learn the squares of all integers from 1 to 20 or 25; pupils should be able to give at sight the square root of such numbers as, 64, 400, 900, 81, 36, 3600.

Square Root by Factoring. The first examples worked in square root should be by factoring. The fact should be emphasized that we seek to find two equal numbers whose product is the given number. For example, $324=2\times2\times9\times9$, or $2^2\times9^2$; therefore the square root of $324=2\times9=18$. Similarly, $2025=5\times5\times9\times9$ or $5^2\times9^2$; therefore the square root of $2025=5\times9=45$. Also, $53361=3\times3\times7\times7\times11\times11$, or $3^2\times7^2\times11^2$; therefore the square root of $53361=3\times7\times11=231$.

This method applies only to perfect squares. If we wished to find the square root of 5329, or the appromixate square root of 3, the desired result could not be conveniently found by factoring.

Other Methods of Extracting Square Root. Occasionally a teacher advocates the teaching of square root as a purely mechanical process, with no attempt whatever to justify the procedure to the mind of the pupil. Such a point of view is contrary to the best educational theories of the day. Pupils should understand each step of the process, but the object is not to enable the pupil to repeat the explanation; it is to justify the various steps to his mind. After this has been done the pupil should be required to work numerous examples until he has acquired facility in the process.

Algebraic Method. This method is applicable to the extraction of any desired root, but no root higher than the third is necessary in arithmetic.

The square of 45 may be found as follows:

$$
\begin{array}{l}
(45)^2=(40+5)^2 \\
\quad 40\ +5 \\
\quad 40\ +5 \\
\hline
\quad 40^2+(40\times5) \\
\qquad +(40\times5)+5^2 \\
\hline
40^2+2(40\times5)+5^2
\end{array}
$$

Similarly, $(97)^2=(90+7)^2$

$$
\begin{array}{l}
\quad 90\ +7 \\
\quad 90\ +7 \\
\hline
\quad 90^2+(90\times7) \\
\qquad +(90\times7)+7^2 \\
\hline
90^2+2(90\times7)+7^2
\end{array}
$$

Every number composed of two or more digits may be regarded as composed of tens and units. Thus, 45 equals 4 tens plus 5 units; 97 equals 9 tens plus 7 units; 434 equals 43 tens plus 4 units. By working several examples similar to the above, we find that the square of any number will equal the square of the tens, plus twice the tens by the units, plus the square of the units. If t represents the number of tens in any given number and u represents the number of units, we may say that the square of the number equals $t^2+2tu+u^2$.

Since the square of the smallest number of one digit contains one digit, and the square of the largest number of one digit contains two digits, the square of any number of one digit must contain either one or two digits. Similarly since the square of 10, the smallest number of two digits, is 100, and the square of 99, the largest number of two digits, is 9801, we conclude that the square of a number of two digits must contain three or four digits. This may be extended to larger numbers. If any integral number is divided into groups of two digits each, from the right to the left, the number of digits in the root will be the same as the number of groups of digits. The last group to the left may contain one or two digits. Thus, the square root of $\overline{18}\ \overline{14}\ \overline{76}$ contains three digits. The square root of $\overline{169}$ contains two digits.

Required to find the square root of 2025.

If 2025 is a perfect square, its square root must contain two digits, since the number itself contains four digits. The largest square in 20 is 16, and the square root of 16 is 4. The tens digit of the root is therefore 4. The square of 4 tens, or 40, is 1600. 425 is composed of two times the tens times the units, plus the square of the units. We do not know the square of

$$\begin{array}{r|l} \underline{4}\ \underline{5} & \\ 2025 & \\ \underline{1600} & \\ 80 & 425 \\ \underline{5} & \underline{425} \\ 85 & \end{array}$$

the units, but we know that it is small compared with two times the tens times the units. We may therefore think of 425 as composed approximately of two factors; one of these factors (two times the tens) is known, and the other is not known. We therefore divide 425 by two times the tens in order to find the units, which is the other factor. In this division we remember that 425 is somewhat larger than the product of these factors, hence we make a slight allowance for this excess. In the problem above, two times the tens equals 2×40, or 80. 80 is contained in 425 five times. The units digit of the root is 5. The 5 is added to the 80 and this result is then multiplied by 5, because we may regard two times the tens times the units plus the square of the units as (two times the tens plus the units) times units.

In practice we may omit the zeros in the square of 40, also the zero of 80. We may then annex the 5 to the 8; see example.

```
      45
     ___ ___
     20  25
     16
   ___________
  85 | 4  25
     | 4  25
     |_______
```

The method may be used in finding the root of any perfect square, or the approximate root of a number not an exact square. If the given number contains more than two groups of two digits each, we may think of the part of the root found as so many tens with reference to the next digit to be found. This should be clear from the example that follows.

Required to find the square root of 107584.

```
      3   2   8
     ___ ___ ___
     10  75  84
      9
   ___________
  62 | 1  75
     | 1  24
   ___________
 648 | 5184
     | 5184
   ___________
```

The square root of 107584 is therefore 328. The rule for grouping the digits of a decimal whose square root is required may be developed by a method similar to the one followed in determining how to group the digits of an integral number. Since the square of tenths is hundredths, and the square of hundredths is ten thousandths, we conclude that in extracting the root of a

number containing a decimal fraction we must begin at the decimal point and group the digits in twos from the left to the right. Thus, $\overline{.57}\ \overline{46}\ \overline{28}$; a zero may be annexed to complete the number of digits in the last group. Thus, $\overline{.38}\ \overline{75}\ \overline{2} = \overline{.38}\ \overline{75}\ \overline{20}$. With the exception of this difference in grouping the digits, the extraction of square root is the same for decimal fractions as for integers. In extracting the square root of decimal fractions, the chief difficulty is encountered in such examples as the following: $\sqrt{.3}$; $\sqrt{.124}$; $\sqrt{1.5}$. The difficulty here is usually due to a mistake in pointing off, or grouping the digits. If a zero is annexed, so that the number contains an even number of decimal places, this difficulty is removed. For example, $\sqrt{.3} = \sqrt{.30}$; $\sqrt{.124} = \sqrt{.1240}$; $\sqrt{1.5} = \sqrt{1.50}$.

Not all numbers are perfect squares, hence not all numbers can be expressed as the product of two equal factors. For example, if we wish to find the square root of 2, we may annex zeros to the right of the decimal point and carry the extraction of the root to any required degree of accuracy. $\sqrt{2} = 1.414$; $\sqrt{3} = 1.732$. The square roots of 2 and of 3 are so frequently used in mensuration that they should be memorized. In practice, numbers that are exact squares are not commonly involved in problems.

If we define the square root of an expression to be one of its two equal factors, it is evident that only abstract numbers can have square roots. No number multiplied by itself equals 9 square feet, or $9, or 9 books. Hence 9 square feet, $9, and 9 books have no square roots.

Geometric Method. The extraction of the square root of 2025 may be explained by the use of a diagram, and this method may be used in any problem in square root. Suppose that $ABCD$ is a square containing 2,025 sq. units. It is required to find the length of one side of this square.

A square whose area is 1,600 square units is 40 units

on each side, and a square whose area is 2,500 square units is 50 units on each side. The side of the square *ABCD* must therefore contain between 40 and 50 units. If a square whose side is 40 units is taken out, then 1,600 square units are taken out and the remaining area is 425 square units. In the figure, *AHRM* is 40 units on each side, and therefore contains 1,600 square units. Since the square *ABCD* contains 2,025 square units, the rectangle *MRFD* plus rectangle *RHBE* plus the square *FREC* contains 425 square inches. The combined length of the two rectangles is known to be 80 units. The problem is to find how wide these rectangles and the square must be so that their combined area shall be 425 square units. If the width were 6 units, the combined area would be more than 6×80 square units, or 480 square units. Try 5 units. The combined area of the rectangles would then be 5×80 square units, or 400 square units, and the area of the square would be 5×5 square units, or 25 square units. The combined area of the rectangles and the square is therefore, 425 square units. This is known to be the area of the remainder of the square *ABCD* after the area of the square *AHRM* has been considered. The length of *AB* is therefore 40 units plus 5 units, or 45 units.

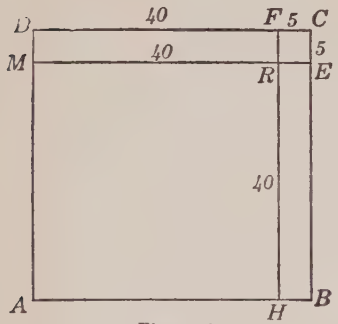

Figure 6

It should be noticed in the above explanation that the number of units in the combined areas of the two rectangles and of the square is the remainder after the number of units in the square *AHRM* has been subtracted. The sum of the number of units in the length of the two rectangles is

the trial divisor, and the number of units in the side of the square *RECF* completes the divisor.

The square root of any perfect square, or the approximate square root of any number not a perfect square, may be found by this method.

Whether square root is taught by the algebraic, or by the geometric method, the pupils should be asked to formulate a statement of the necessary steps in the process.

Such a statement as the following is valuable as a working rule. 1. Separate the number into periods of two digits each, beginning at the decimal point. 2. Find the greatest square in the left-hand period and subtract it, bringing down the next period. 3. Divide the remainder by twice the part already found. 4. To this divisor add the number thus found and multiply this sum by the number found. 5. Subtract this result and bring down the next period. Proceed as in steps 3 and 4.

Estimating the Square Root of a Number. Since the square root of a number is one of the two equal factors of the number, finding a square root may be thought of as a problem in which the dividend is given and the divisor and quotient are known to be equal, but both are to be found.

A widely used and valuable method for finding square roots may be explained as follows.

Suppose the square root of 32 is required.

It is evident that the square root of 32 is more than 5 and less than 6, since $5\times5=25$ and $6\times6=36$. Assume the root to be 5.6 since it is nearer to 6 than to 5. Dividing 32 by 5.6 we get a quotient of 5.714. We now know that the square root of 32 must be between 5.6 and 5.714. Taking half the sum of 5.6 and 5.714 we get 5.657 which we now divide into 32 to determine whether it is the required root. The quotient is found to be 5.657. Therefore, the square root of 32 is 5.657.

This method for finding square root may be summarized as follows:

First estimate the square root. Divide the number whose square root is to be found by the estimated square root. Take half the sum of the estimated square root and the quotient just found and use this half sum as the second estimate of the square root. Divide the number whose square root is to be found by this second estimated square root and continue the process as before. The process may be continued until the square root has been found to any required degree of accuracy. The square root always lies between the last divisor and the last quotient.

Apply this method to the finding of the square roots of each of the following: 17 21 58 743.

Square Root of Fractions. The square root of a common fraction, both of whose terms are perfect squares, may best be found by extracting the square root of both terms.

Thus: $\sqrt{\frac{4}{9}} = \frac{2}{3}$; $\sqrt{\frac{169}{225}} = \frac{13}{15}$.

If the terms are not perfect squares, the square root may be found in either of two ways: (*a*) by reducing the fraction to a decimal, and then extracting the root to the required number of decimal places; or (*b*) by multiplying both terms of the fraction by a number that will make the denominator a perfect square, and then proceeding as in the Method B below.

Method A: $\sqrt{\frac{2}{5}} = \sqrt{.4} = .632$

$\sqrt{\frac{3}{7}} = \sqrt{.428571} = .654$

Method B: $\sqrt{\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{1}{5}\sqrt{10} = \frac{3.162}{5} = .632$

$\sqrt{\frac{3}{7}} = \sqrt{\frac{21}{49}} = \frac{1}{7}\sqrt{21} = \frac{4.582}{7} = .654$

Applications of Square Root. You will find a number of important applications of square root in the subject of Mensuration. Many workmen employed in the building trades use square root in practical ways. Find out what the builder means by "squaring a corner" and show how square root is used.

The following questions and problems will suggest other practical applications.

1. Can a carpenter square a corner by using a right triangle whose sides are 6 inches, 8 inches, and 10 inches?

2. Carpenters sometimes find the pitch of a roof by using a carpenter's square and a rule. Ask some carpenter to show you how this is done.

3. A pole whose top is 24 feet above the ground is to be braced by a wire which is to be attached 4 feet from the top of the pole and 15 feet from the foot of the pole. Allowing $1\frac{1}{2}$ feet for attaching, how long must the wire be?

4. A rectangular lot is 60 feet by 150 feet. Compare the diagonal with the sum of the length of the two sides.

SUGGESTED READINGS

"Roots by inspection." *Scientific American*, Vol. 108, p. 354.

CHAPTER XXX

MENSURATION

Mensuration affords the teacher the opportunity to open the way to new and inviting fields in mathematics. The subject is justified because it is immediately useful. It is taught primarily to give facts and develop skill related to the affairs of everyday life.

Method to Be Employed. No comprehensive notion of the interpretative value of mensuration is possible unless its truths are demonstrated. By the time the subject is introduced in the seventh or eighth grade, the children already have a fairly liberal notion of its elementary phases. They have learned in the lower grades that mensuration deals with lines, surfaces, and volumes. It remains to apply the ideas thus acquired in the earlier grades to situations of great utility.

Measurement of Plane Figures. In practice, areas and volumes are commonly found by indirect measurements. The surveyor and the astronomer constantly resort to indirect measurements. Such measurements are used to determine the heights of mountains, the widths of rivers, and the distance to the sun, moon and stars.

The length of a straight line may be found and often is found by applying the unit of measure directly to the line. The fact that of all plane figures the rectangle is the only one whose area may be found directly by applying the unit of measure (a square) and then counting the number of times it is applied, should be developed and emphasized. In actual practice the area of a rectangle is not found by

direct application of the unit of area. The area of a floor or of a garden is never so found.

The rectangle is the plane figure from which the mensuration of all other polygons is developed. The area of rectangles, having been studied in the earlier grades, should be reviewed as a basis for the development of the mensuration of other polygons. Show that a square one unit long and one unit wide contains one square unit, and that a rectangle two units long and one wide contains two square units. Many teachers still permit their pupils to say the area of a rectangle 2 inches long and 1 inch wide is 2 inches $\times 1$ inch $= 2$ square inches. The area of a rectangle whose length is 4 feet and whose altitude is 3 feet should be expressed as $4 \times 3 \times 1$ sq. ft., or as 3×4 sq. ft. Such a statement gives the correct result, and does not violate any of the principles of multiplication. If the length and width are expressed in inches the area will be expressed in square inches; if in feet, the area will be expressed in square feet. The number of units in the length of a rectangle shows how many squares may be placed in one row and the number of units in the width shows the number of such rows. If the work is properly developed, pupils will not get the idea that feet times feet gives square feet.

Enough class exercises should be given to insure a clear notion of area. Pupils may be required to estimate the number of square inches or square feet in a book cover, a window, a picture, a blackboard, and then to test the accuracy of each estimate. The ability to estimate areas in square inches, square feet, and square yards may be developed by cutting, drawing, estimating, and verifying of square rods and acres by actual measurement. Later practical problems involving area should be given. Perhaps the best of these deal with land measure and surveying, flooring, carpentering, lathing, plastering, and papering.

The Parallelogram. After the pupil is familiar with the

mensuration of the rectangle, he should be taught that the area of any parallelogram is equivalent to that of a rectangle of the same base and altitude as the parallelogram. This may be deduced by a study of the figures. If we cut out the triangle MBC and put it in the position of triangle HAD, we shall have a rectangle. This rectangle is evidently composed of the same parts as the original parallelogram and is, therefore, equal to it in area. Since the rectangle and parallelogram have bases of equal length and also have equal altitudes, the area of the parallelogram may be stated in terms of the rectangle. It should be noted, however, that the area of a general parallelogram cannot be found by applying the square unit of measure, because the square unit will not exactly fit at the corners. At this stage of the work, the pupil can only infer that the figure is the equivalent of a rectangle, because it appears to be so. The teacher should not overlook the opportunity to tell the pupil that when he learns geometry he will be able to prove that the area of the rectangle is equal to that of the parallelogram.

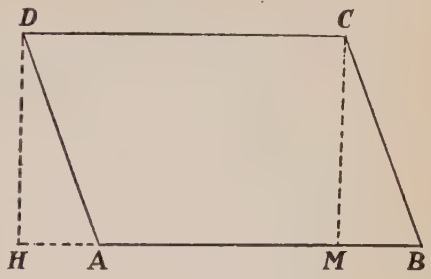

Figure 7

Measurement of Triangles. The manner in which the various kinds of triangles are related in area to parallelograms and rectangles should be developed. Before this is done, the attention of pupils should be called to the classification of triangles on the basis of the length of their sides or the size of the angles. The essential differences in triangles may be discovered by the pupils themselves. An

equilateral triangle has all of its sides equal. An isosceles triangle has two of its sides equal. A scalene triangle has no two sides equal. Triangles are also called acute-angled, obtuse-angled, or right-angled, the classification depending upon the size of the largest angle.

In order to emphasize and fix these conceptions such questions as the following may be asked. Is an isosceles triangle necessarily equilateral? May an isosceles triangle be equilateral? Which is the more general term, equilateral or isosceles? May a right triangle be isosceles? May it be equilateral? May an obtuse-angled triangle be equilateral or isosceles?

The simplest case showing the relation of a triangle to the rectangle is the half square. It seems evident to a pupil that a diagonal divides the rectangle into halves. This fact can be shown by folding a square of paper.

It can also be shown that a triangle has half the area of a parallelogram having the same or equal base and altitude as the triangle. By numerous examples similar to these the formula for finding the area of a triangle may be developed.

The area of a triangle is one-half the product of its base and altitude.

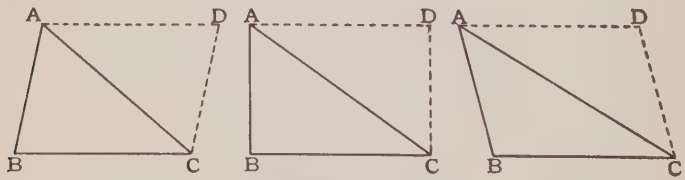

Figure 8

Measurement of Trapezoid and Trapezium. The trapezoid is often encountered in practical mathematical situations. The method of finding the area of a trapezoid may be developed in either of two ways. The points *O*

and E are the midpoints on the lines DA and CB between the parallels DC and AB.

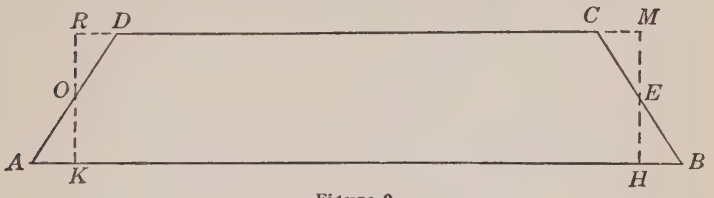

Figure 9

By cutting off the triangle BHE and placing it in the position CME, and placing the triangle AOK in the position ROD, the result is the rectangle $KHMR$. The altitude of the trapezoid has not been altered and the sum of the bases of the original trapezoid $ABCD$ is equal to the sum of the bases of the resulting rectangle.

Figure 10

The same result may be derived by constructing the trapezoid B equal to the trapezoid A. The result is a parallelogram. The altitude of the parallelogram is the same as that of the trapezoid, and half of the sum of the bases of the original trapezoid is equal to one base of the parallelogram. The area of the trapezoid can now be stated since the area of the parallelogram is known.

As early as 1500 B. C. the Babylonians had discovered how to find the area of the trapezoid.

Measurement of Circles. The area of a circle is important because it is needed in finding the volume of a cylinder. There are three formulas for finding the area after the method commonly found in most good texts has somewhat

justified the process to the pupil's mind. Since, as is explained in the textbooks, the number of units in the area of a circle is equal to one-half the number of units in the radius, times the number of units in the circumference, we have the formula $A=\frac{CR}{2}$, where A, C, and R represent the number of units in area, circumference, and radius respectively. From this formula the other two formulas for the area of the circle may be easily deduced. These formulas are: $A=\pi R^2$ and $A=\frac{\pi D^2}{4}$. If the teacher will have her pupils carefully measure the circumferences and diameters of several rather large circles of different sizes (a plate, a hoop, or any circular object whose dimensions can readily be found, will serve the purpose), and then have the pupils divide the length of the circumference by that of the diameter, the quotient in every case will be found to be about $3\frac{1}{7}$. This ratio of the length of the circumference to that of the diameter is proved in geometry to be always the same, and to be approximately equal to 3.14159. For all practical school purposes the values, 3.1416 or $\frac{22}{7}$, are sufficiently accurate.

The Ahmes papyrus gives the value of π as $(\frac{16}{9})^2$ or 3.1605. The Hebrews and Romans considered π equal to 3. The value of π was computed to 707 decimal places by William Shank. It is said that the symbol π was first suggested in 1706.

Since the ratio of the length of the circumference to that of the diameter is constant, a special symbol is used to represent this value. This symbol is the Greek letter *pi*. We say therefore $\frac{C}{D}=\pi$, or $\frac{C}{2R}=\pi$, since $D=2R$, or $C=2\pi R$.

If we substitute for C its value $2\pi R$ in the formula $A=\frac{CR}{2}$

we shall have $A = \frac{2\pi R \times R}{2}$, or πR^2. Since $D = 2R$, therefore $R = \frac{D}{2}$ and $R^2 = \frac{D^2}{4}$. If we put this value for R^2 in the formula πR^2, we shall have $A = \pi \frac{D^2}{4}$

Since $\pi = 3.1416$, we have $A = \frac{3.1416 D^2}{4}$ or $.7854 D^2$

The pupil should know these formulas for area:

$$\frac{CR}{2} \qquad \pi R^2 \qquad \frac{\pi D^2}{4}$$

Measurement of Rectangular Solids. Just as the area of most plane figures may be found by comparing them with the area of a rectangle, so we may find the volume of regular solids by comparing them directly or indirectly with the volume of the rectangular solid. The rectangle is the only plane figure whose area can be determined by applying the square unit of measure and counting the number of times it is contained in the rectangle whose area is desired, and the rectangular solid is the only solid whose volume can be found by applying the cubic unit of measure and counting the number of times it is contained in the solid whose volume is required. The number of cubic units that can be placed in one layer on the bottom of a box is found by multiplying the number of units in the length by the number in the width. The number of units in the depth of the box shows the number of such layers.

If the teacher has small cubes and will build up before the class, or have some pupil build up, rectangular solids of various dimensions, the fact that the number of units in the volume of a rectangular solid is equal to the product of the number of units in its three dimensions can be seen.

It is desirable that the teacher should not permit the statement which is so commonly used in such problems, 4 in.$\times$3 in.$\times$2 in.$=$24 cu. inches.

The Use of Models. Space does not permit an extended discussion of the various methods whereby the volumes of the various solids, commonly considered in the arithmetic of the grades, may be shown to depend directly or indirectly upon the volume of the rectangular solid. Most arithmetics contain a sufficiently detailed discussion of these facts. The best interests of the class will frequently be served, if the pupils have access to models of the various solids that are to be studied. Inexpensive models for this purpose can usually be made. Many of the more common solids can be easily made out of wood or pasteboard by the teacher or the pupils. In studying the pyramid, the cylinder, the cone, and the sphere it is very desirable to have models at hand, as they will frequently make clear some difficulty due to the inability of the pupil to image properly the solid under consideration. That models may be used to excess in the study of mensuration should not condemn the use of models, but should argue for more wisdom in their use. It is wise to dispense with the use of objects in mensuration just as soon as the pupil can image properly the figure studied. Some pupils can image much more readily and more distinctly than others, and the teacher should be careful to use the objective material only in those cases where it seems needed.

Measurement of Cylinder. It is desirable in studying the lateral area of a cylinder to have the pupil think of a piece of paper so cut that it will just cover the lateral surface when wrapped about it. When the paper is removed, the pupil will readily see that it is in the form of a rectangle. The base of the rectangle is equal in length to the circumference of the cylinder, and the altitude of the rectangle is equal to the altitude of the cylinder. Since

the number of units in the area of the rectangle can be found, the number of units in the lateral surface of the cylinder can readily be determined. Let the pupil try to image the shape of the paper that could be made to just cover the lateral surface of a right cone. He should see that the paper is in the form of a sector of a circle. From this the formula for the lateral surface of a cone may be determined.

Measurement of Pyramids and Prisms. For comparing the volume or cubical contents of a right pyramid with that of a right prism of equal base and altitude it is well to fill the pyramid with some substance such as sand or water and pour it into the prism. When the pupil finds that the pyramid must be filled three times and the contents poured into the prism in order that the latter may be full, he is ready to infer, as is proved in geometry, that the volume of a pyramid is equal to one-third of the volume of a right prism of equal base and altitude. The volume of a cone may be compared with that of a cylinder of equal base and altitude by the same method.

Situations do not often arise which require one to find the volume of a pyramid, cone, or sphere.

Measurement of the Sphere. Pupils are usually interested in the experiment to determine the area of the surface of a sphere. Cut a wooden ball by a plane through the center. Place a tack at the center of the sphere and let a pupil wind a firm cord about this, sufficient to just cover the surface of the circle. Then let the pupil wind sufficient of the same cord about the hemisphere to just cover its curved surface, and compare the amount of cord required to cover the two areas. If the work has been carefully done, it will be found that twice as much cord is required to cover the sphere as to cover the plane surface of the hemisphere. The number of units in the area of the circle of the sphere is known to be πR^2, where R repre-

sents the number of units in the radius—therefore, the number of units in the curved surface of the hemisphere is $2\pi R^2$ and the surface of the entire sphere is $4\pi R^2$.

Use of Literal Representation. Most pupils see the advantage of letting s represent the number of units in the side and a the number of units in the altitude. If the pupils are required to use these and other abbreviations in the work of the upper grades, a large saving of time will result.

Problems

It is now proposed to consider a few problems in mensuration, somewhat more difficult than most of those in the arithmetic of the grades. The problems that follow are of a type that are a good test of the ability of the pupil to image clearly.

1. Compare the area of a square with that of the largest circle that can be cut from it.

In solving such a problem it is necessary that the pupil should image the figure clearly, and should see what dimensions the circle and the square have in common. If the pupil is not able to image this relationship readily the figure may be drawn. It is seen that the side of the square is equal in length to the diameter of the circle.

Let R = the number of units in the radius of the circle.

Then $2R$ = the number of units in the diameter of the circle.

Therefore, $2R$ = the number of units in the side of the square.

The area of the circle is πR^2.

The area of the square is $(2R)^2$ or $4R^2$.

Therefore the area of the circle is to the area of the square as

$$\pi R^2 \text{ is to } 4R^2 \text{ or } \frac{\pi R^2}{4R^2} = \frac{\pi}{4}$$

$$\text{Since } \pi = \frac{22}{7} \text{ we have } \frac{\left(\frac{22}{7}\right)}{4} = \frac{22}{28} \text{ or } \tfrac{11}{14}.$$

Therefore, the area of the circle is $\frac{11}{14}$ of the area of the square.

2. The problem to compare the area of a circle with the largest possible square that can be cut from it, is slightly more difficult. In this problem the diameter of the circle is seen to be equal to the diagonal of the square.

Let R = the number of units in the radius of the circle.

Then $2R$ = the number of units in the diagonal of the square.

Since the number of units in the diagonal of the square $= 2R$, the number of units in the square of one side is easily found to be $2R^2$. This is the area of the square.

The area of the circle is πR^2.

The area of the circle is to the area of the square as

$$\frac{\pi R^2}{2R^2} = \frac{(\frac{22}{7})}{2} = \tfrac{22}{14} = \tfrac{11}{7}.$$

The fact that such problems are considered also in geometry should not deter the teacher from considering a few of them in the more advanced work in mensuration in the grades.

3. Compare the volume of a cube with the volume of the largest possible sphere that can be cut from the cube. It is easily seen that the length of the diameter of the sphere is equal to the length of the edge of the cube.

Let R = the number of units in the radius of the sphere.

Therefore $\frac{4\pi R^3}{3}$ = the number of units in the volume of the sphere.

$2R$ = the number of units in the edge of the cube.

Therefore $8R^3$ = the number of units in the volume of the cube.

The ratio of the volume of the cube to the volume of the sphere is therefore, $\frac{8R^3}{\frac{4\pi R^3}{3}} = \frac{6}{\pi} = \frac{6}{\frac{22}{7}} = \frac{42}{22} = \frac{21}{11}$.

The volume of the cube is, therefore, $\frac{21}{11}$ of the volume of the sphere. It would be an easy problem to compare the surface of the cube with that of the sphere.

It is sometimes well to put the best pupils of the class on their mettle, and such problems are well adapted to this purpose.

Observational Geometry. There is a tendency to curtail the work in mensuration, which formerly consisted largely of definitions, formulas, and problems, and to include some elementary geometry in the course of the seventh and eighth grades. The type of work introduced is variously known as constructional, observational, inventional, intuitional, or concrete geometry. Its purpose is to acquaint the pupil with the more important geometrical concepts, to train the eye and the hand in the use of the straight edge, compasses, triangle, and protractor, to develop the powers of observation and intuition as applied to geometrical forms, and to enable the pupil to master certain fundamental concepts which have frequent practical applications in life.

It is desirable that pupils who do not enter the high school should have some knowledge of the fundamental concepts of geometry such as may readily be secured. An introductory course in intuitive geometry prepares those pupils who enter high school for the more formal study of the subject.

Some work in introductory geometry is now given in

the grades in most of the best schools of the country. The instruction is sometimes made an integral part of the course in arithmetic, and is included under the topic of mensuration. In some schools one or more periods a week are regularly devoted to the work in the seventh and eighth grades and the connection with mensuration is not so immediate.

In many of the European schools the teacher of mathematics is also the teacher of drawing, and work of the kind referred to above is made a part of the instruction in drawing. It has been suggested that one reason for the slowness of the introduction of this type of work into American schools is that some work in mechanical drawing is usually given in the art courses. When such has been the case, emphasis has usually been placed upon artistic results rather than upon the training of the power of observation and generalization, because our art teachers do not usually emphasize the subject from the point of view of mathematics. The schools in which these constructional exercises are taught usually attempt to correlate the work as closely as seems practicable with the course in manual training.[1]

In several of the European countries the course in arithmetic includes the study of some of the simple geometric forms and constructions as early as the fifth and sixth school years. The pupils become familiar with the fundamental properties of straight lines, angles, triangles, quadrilaterals, polygons, and circles, and models of some of the simpler geometrical solids are constructed.

Simple and easily available material for the study of elementary geometrical forms is abundant, and the course in mathematics in the grades may be both enriched and

[1] See *Mathematics in Elem. Sch. of U. S.* Report to International Commission, p. 136.

vitalized by a judicious use of this material. Many of the most important theorems of geometry may be intuitively established by the use of constructions and measurement. A better insight is also gotten into most of the rules and formulas of mensuration.[2]

Pupils of the grammar grades should be taught the meaning of such terms as perpendicular, right angle, parallel, etc. They should be required to construct at the blackboard, and by the use of paper or cardboard, the right, isosceles, equilateral and scalene triangles, and by means of cutting out, tracing, applying, and the use of the protractor they should discover experimentally the elementary truths about congruency of triangles and parallelograms, and the size and relation of angles in the various triangles. The formulas for the area of the triangles, parallelogram, and trapezoid should be discovered through drawing and paper-cutting or by means of tracing. In some schools plotting paper is extensively used as a means of estimating areas.

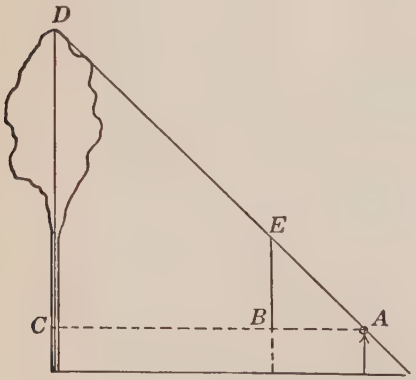

Figure 11

Congruent and Similar Figures. The properties of congruent and of similar figures may be used by the pupils in a variety of interesting and instructive ways. Ask the pupils to devise two or three ways for determining the height of a tree or a building. The following methods are suggested: (*a*) By com-

[2] See Wentworth, Smith, Brown, Junior High School Mathematics, published by Ginn and Company of Boston.

paring the length of the shadow of a tree or building with that of a pole of known height. (*b*) By use of an isosceles right triangle. If $AB = BE$, then $AC = CD$, therefore to find CD, measure AC. (*c*) By drawing to scale.

Let pupils determine the distance between two inaccessible points by use of congruent triangles.

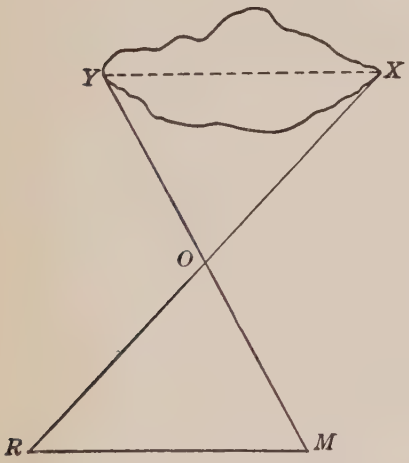

Figure 12

Set up a pole at O and measure the distances OX and OY. On a line with X and O put a stake at R such that the distance $OX = OR$. On a line with Y and O put a stake at M such that $OY = OM$. Measure the distance RM. This will be equal to XY, which is the required distance.

Ask the pupils to devise methods of determining the width of a river without crossing it.

The following method may be used:

First Method. To determine the distance AB.

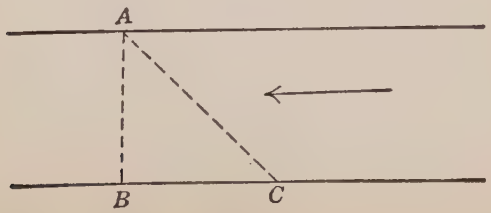

Figure 13

A pupil who has an isosceles right triangle may walk along the bank sighting along the hypotenuse AC and holding one arm of the triangle parallel to the line BC, until he reaches a point C, such that the point A on the opposite bank from B is seen. The distance BC is then equal to AB.

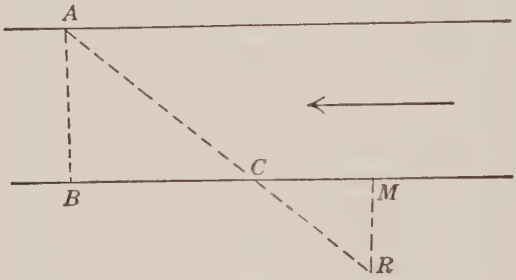

Figure 14

Second Method. To determine the length of AB.

Mark off a line BM perpendicular to AB and place stakes at points B and M. Erect perpendicular MR of a convenient length to line MB. Place the stake at C where the line BM intersects the imaginary line AR. The distance AB is then found from the proportion $\frac{AB}{MR} = \frac{BC}{MC}$, in which MR, BC and MC are known.

CHAPTER XXXI

GRAPHS

Newspapers, magazines, trade journals, and many official reports make frequent use of graphs in order to make clear the relations among related quantities. It is desirable that instruction in this subject should be given in the schools. The pupil of the seventh or eighth grade is more or less familiar with the representation of the relative sizes of armies and navies by men of various heights. He has seen lines used to represent the growth of the population, and he knows that curves are sometimes used to represent variations in temperature or fluctuations in the prices of certain commodities. In all of the more progressive European countries the subject of graphs is systematically taught before the pupil enters what corresponds to our secondary school. Teachers in this country are beginning to appreciate the fact that the graph may be used to advantage to illuminate certain topics and to emphasize the relations between various magnitudes.

The graph is an effective method of impressing upon the pupil relations that might otherwise be obscure. The underlying principles are easily understood, and some consideration should be given to the subject in the grades. Rates of increase and decrease are always shown more clearly by the use of graphs than by tables of statistics. When two or more graphs are drawn on the same scale and with the same axes, comparison of the variations can be readily made. The graph should not be taught as an end in itself, but as a means to an end. It should be used to supplement, not to supplant analyses. In order to make

a graph we must have a series of number facts which can be compared.

When magnitudes are so related that any variation of one causes a variation of the other, each is said to be a function of the other. For example, the distance that a train will run in a certain time is a function of the rate, because the distance varies as the rate varies. The interest at a given rate is a function of the principal and of the time, because as these vary, the interest varies. The area of a triangle is a function of its base and its altitude, and the area of a circle is a function of its radius.

Walter's record on a series of five tests, in each of which he tried to do 10 addition examples within a certain time limit, was as follows: On the first trial he got 6 correct answers out of 10, on the second, 8, on the third, 7, on the fourth, 8, and on the fifth, 9.

Walter's record is shown by the diagram. By means of such a graph Walter can at once note the progress that he made throughout the series of five tests.

As shown by this graph, the figures at the bottom indicate the number of trials. Those along the left side indicate the number of examples worked correctly.

By observing this graph, can you tell how many examples Walter solved correctly on his third trial? On his second trial? On his fourth trial? Can you show

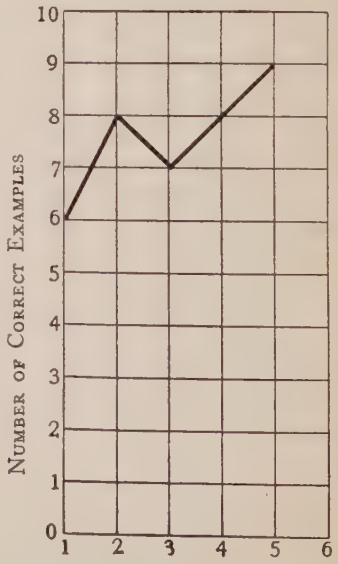

Figure 15

how this graph would have appeared if Walter's record on the five tests had been 5, 7, 5, 8, 9?

Pupils will find it interesting and profitable to make a graph of their records. These graphs should be kept so that they may note the progress made from time to time throughout the year.

Most pupils like to graph the daily variations in temperature. Interesting graphs may be made showing the fluctuations in temperature at a given hour — for example, 9 A. M. — for several consecutive days. The graph below shows this.

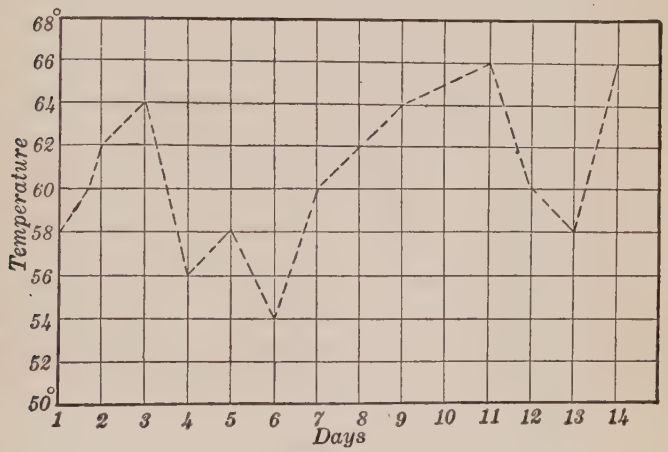

Figure 16

At 9 A. M. on the first day the temperature was 58°; on the sixth day it was 54°; on the eleventh day it was 66°.

We may show the growth in population of a state, county, or city; the increase or decrease in crop production over a period of years; the changes in price of a given article; or something else that offers numbers for comparison. In making such graphs round numbers are commonly used.

Pupils may be asked to graph the growth in population of the city, county, or state in which they live, and then, using the same axes and the same scale, graph the growth in school population for the city, county, or state. Pupils will be interested to compare these graphs. Similar graphs may be made for the nation. Statistics for such graphs may be readily secured in most communities.

Numerous interesting facts whose relations are emphasized by the use of graphs may be easily found by teacher and pupils. Most pupils from rural communities will be interested to graph the yearly yield of wheat, corn, oats, barley, potatoes, cotton, sugar beets, or rice.

County, state, and government reports furnish numerous statistics that are of interest to pupils.

If the school engages in athletic contests, some of the pupils who are interested in these activities may be asked to graph the results of a series of contests. The number of points or of goals scored by each player or each team may be graphed.

Some pupils in the school may be interested in keeping a graphic record of the attendance, and to compare this graph with the graph of various facts that affect the attendance, such as temperature, rainy days, number of cases of sickness in the community, etc.

Ask the pupils to make a graph for the simple interest on $1 at various rates, and require them to use this graph in solving several problems. Compare the graph for simple and for compound interest on a given principal, at a given rate, for 6, 7, or 8 years.

The Bar Graph. The bar graph, one of the most common forms of graphic representation, is extensively used because it is easy to construct and to interpret. Since the graph represents the size of a number by the length of a line, only one dimension is to be considered.

The bar graph below shows the population of the United

States in each census year from 1880 to 1920 inclusive, and so enables one to note at a glance the increase in population which took place during the period indicated. Each horizontal division on the squared paper represents 5 million. As is usual, round numbers are used.

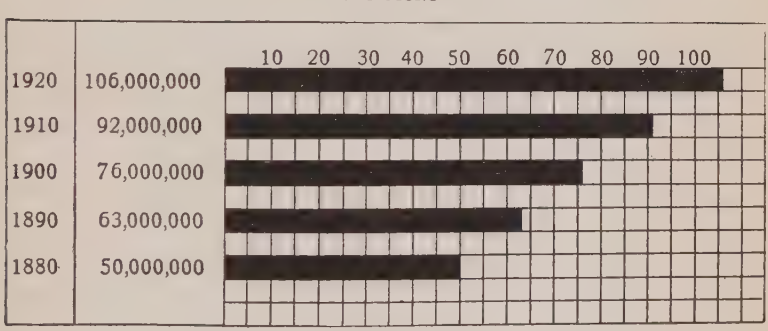

Figure 17

As shown by the graph, what was the population for each census year? What does the graph tell about the growth of population from 1880 to 1920? How does each ten years' increase compare with the increase of the preceding ten years? What other facts about the growth of our population does this graph show?

Before constructing a bar graph, ask yourself the following questions.

1. What is the largest number to be represented?
2. How many spaces are available to represent this number?
3. What value can each space most conveniently represent?

Often a bar graph is constructed on a single bar, as shown below. In such cases, each division of the graph is either given a different color, or is cross-hatched, as shown

here. The figure below shows the sales made during one month by each department of a large department store.

The Circle Graph. The circle graph, like the more common single-bar graph is used to show the relationship of parts to a whole. Each part is represented by a sector of the circle. Sometimes different colors or hatchings are applied to the sectors to make the contrasts among them more striking.

The circle graph is not easily constructed, because it is difficult to measure accurately or estimate the size of the angle or the length of the arc of the sectors. When well made, however, this graph is an excellent method of showing relationships.

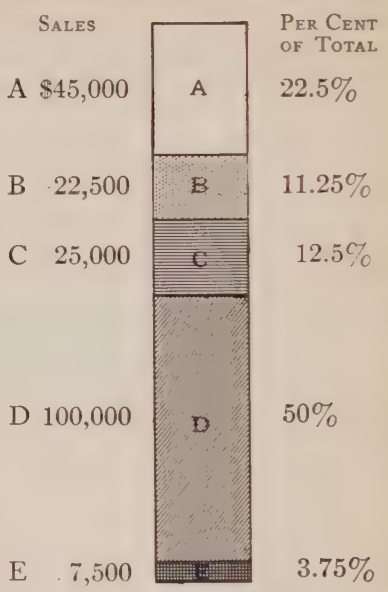

Figure 18

In your geography, science book, or a magazine, find examples of the circle graph.

This circle graph which shows the amount of cotton produced in one year by the leading cotton-producing countries of the world, was constructed as follows.

The total amount of cotton produced was 19,595,000 bales. Of this amount the United States produced 13,366,000 bales; India, 2,976,000 bales; Egypt, 1,251,000 bales; and the rest of the world 2,002,000 bales. Of the total, the United States produced 68.2%; India, 15.2%; Egypt, 6.4%; and the rest of the world 10.2%. Since there are 360 degrees in a circle, we lay off 68.2% of 360

degrees, or 245.5 degrees, to represent the amount produced by the United States. In like manner we lay off 15.2% of 360 degrees, or 54.7 degrees for India, 6.4% of 360 degrees or 23 degrees for Egypt, and 10.2% of 360 degrees or 36.7 degrees for the rest of the world.

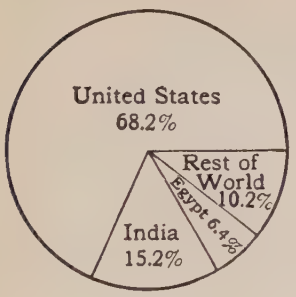

Figure 19

Having indicated the respective points on the circle by means of a protractor, draw a radius from each point to the center of the circle. The resulting sectors will indicate graphically the number facts under consideration.

The Pictorial Graph. The pictorial graph uses pictures of the objects to be compared. The respective pictures vary in size as the number facts vary. Although such graphs are difficult to make, they are very effective for

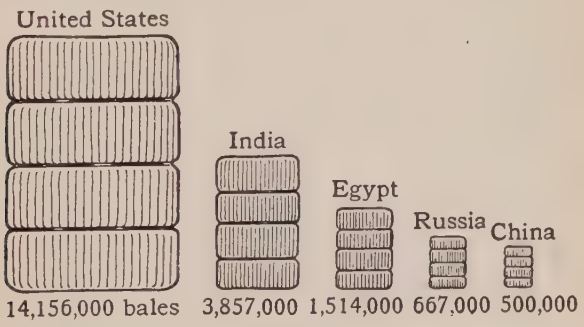

Figure 20

strikingly presenting certain facts. The value of pictorial graphs is greatly increased when the facts on which they are based accompany the pictures.

The above is a pictorial graph showing the relative

amounts of cotton produced by the United States, India, Egypt, Russia, and China during a recent year.

Suggest how you might use a pictorial graph to represent the number of automobiles used in two states.

Suggest a way to use the pictorial graph for representing the number of pupils in each class in your school building.

Suggest a pictorial graph to represent the following.

1. The sizes of several ocean vessels.
2. The corn or cotton crops of two or more states.
3. The number of fires in a city during each of two years.

Ask the pupils to find in magazines and newspapers examples of the pictorial graph. Whatever graphs they find should be brought to class and interpreted to their classmates.

Suggested Data for Graphs

1. Graph the monthly attendance record of the class or school.
2. Graph the temperature for a certain hour for five consecutive days.
3. Have pupils graph their daily, weekly, or monthly averages in some subject.
4. Graph the growth in population of a city, county, state, or of the United States.
5. Graph the number of bushels, tons, or bales of various crops.
6. Use a bar graph to show the size of each of several states.
7. Graph the weights or ages of the pupils in a class.
8. Graph the variation in price of some common articles of food, such as eggs, potatoes, or butter.
9. Graph the lengths of several of the longest rivers and the heights of several mountains.
10. Show by a circle graph the number of pupils in each grade in a school building.

11. Show by a graph the distance from your city to five of the largest cities of your state.

12. Suggest how a pictorial graph might represent some important facts of interest.

SUGGESTED READINGS

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CHAPTER XXXII

LITERAL ARITHMETIC AND ALGEBRA

There is a marked tendency in this country towards the introduction of some form of literal arithmetic, or of algebra, in the last two years of the grammar grades. In some schools the work in arithmetic is finished by the end of the seventh grade or by the end of the first half of the eighth, and the remaining time is devoted largely to algebra. In other schools some work in algebra is introduced into the seventh or eighth grades without dropping the arithmetic, and with but little effort to relate the two subjects closely. In some schools an effort is made to introduce literal arithmetic or algebra, as opportunity permits, while arithmetic is being studied, and an effort is made to establish a vital connection between the subjects. This last manner of introducing algebra is being rapidly extended and adopted.

In many schools algebra is now taught in some form in the grades. Several arguments have been advanced to justify this practice. It is stated that a proper introduction of the subject interests the pupils to such an extent that they have a greater desire to enter the high school. Some advocate its introduction in order to make the transition from grammar school to high-school subjects more gradual. Others introduce the simple equation, the graph, and algebraic formulas into the work in arithmetic in order to enrich the course and to simplify the solution of certain types of problems.

The Present Tendency. The present tendency is toward the introduction of some work in algebraic formulas, simple equations, and graphs in connection with the

regular work in arithmetic. The traditional lines between arithmetic and algebra are disappearing, and the subjects are being more closely correlated. Many pupils of the upper grammar grades do not enter the high school, and these pupils should have the opportunity to learn the meaning and the use of algebraic formulas and graphs, now so frequently used in trade journals and mechanics' handbooks, and to appreciate the value of the simple equation as a tool in the solution of problems.

One of the chief objections that has been urged against algebra, as it is usually taught in the grammar schools, is that it is an attempt to complete a definite portion of the work of the first year of the high school. There is often too much emphasis upon definitions, theory, and complicated processes. A pupil who does not enter the high school derives but little benefit from a formal study of the four fundamental operations and factoring in algebra.

In the latter part of the sixth, and in the seventh and eighth grades, the teacher should not hesitate to introduce letters to represent numbers. It is no more unnatural, after a short time, for the pupil to use C to represent the cost, I to represent interest, V to represent volume, and N to represent a number, than for him to use N. Y. to represent New York, or A. M. to indicate time before noon. Pupils soon appreciate the fact that calculations involving letters are often simpler than those involving figures. As Young says, "The use of letters to represent numbers simplifies the treatment of some types of problems, which otherwise tend to confuse by mere verbiage."[1] If only those processes that are actually needed in the solution of the problems of arithmetic are emphasized, but little objection will be offered to the introduction of such work in the grades. The transformations that are necessary

[1] Young — *The Teaching of Mathematics*, p. 243.

in the solution of simple equations, and in the evaluation of simple algebraic formulas, can be readily understood by pupils of the seventh and eighth grades.

Strong protests are often made when any innovations in subject matter or symbols are suggested. Smith has pointed out the fact that when it was proposed to write 4×5 instead of *4 times 5*, strong protests were made on the ground that 4×5 was more abstract than *4 times 5* and that it would be best to let well enough alone.[2]

The pupil should recognize the symbols of algebra as a shorthand method of indicating quantities. Arithmetic does not become algebra by the mere use of letters instead of numbers. The solution of many of the problems of common and decimal fractions, of ratio and proportion, of percentage and mensuration, is simplified and abridged by the use of letters for the magnitudes which they represent.

The pupil should have a good deal of practice in the translation of words into the literal notation, before he attempts the solution of many problems involving the new notation. The literal notation is a new language to the pupil, and he cannot master it in one lesson.

The pupil who understands the following forms is prepared to understand the fundamental principles of simple equations.

$4+?=6$	$?+3=7$	$3\times?=12$	$?\times4=12$
$8-?=5$	$?-5=4$	$12\div?=6$	$?\div4=8$

If, instead of the interrogation point, the symbol n is used to represent the required number, its value may be readily found. If teachers will refrain from unnecessary definitions and emphasis upon technical and non-essential points, they will find that pupils like to use the literal notation. In this, as in other subjects, teachers often befog and

[2] Smith — *The Teaching of Arithmetic*, pp. 72-73.

obscure what is otherwise clear to the pupil by attempting to give minute explanations. The literal equation is now quite generally recognized by teachers of the upper grades as of great value in simplifying many arithmetical procedures.

The pupil should appreciate the fact that an algebraic formula is always an abridged and concise method of stating a fact, and he should be able to translate a formula into words, and to translate a simple statement of number facts into a formula. If l represents the length of a field in rods and w represents the width in rods, what does $lw = 1600$ state? If v represents the volume of a sphere and r represents its radius, what does $v = \frac{4}{3}\pi r^3$ state? Such expressions as those above should be as intelligible to the pupil when stated by the use of letters as when stated in words. The literal statement of a problem is more compact than the statement in words, and the relations between the various factors involved may usually be more clearly seen. The sense of correctness of results often fails when working with literal symbols.

The value of the equation as a tool in the solution of problems may be impressed upon the pupil by requiring him to solve some of the problems of the text without the use of letters, and then comparing with these the solutions of the same problems in which literal notation and the simple equation have been employed. Such comparisons are often a revelation to a pupil.

Note.—Teachers who wish to study further the present trend towards literal arithmetic and algebra are referred to the Wentworth, Smith, Brown, Junior High School Mathematics series, published by Ginn and Company, Boston.

SUGGESTED READINGS

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CHAPTER XXXIII

PRESENT TENDENCIES IN ARITHMETIC

It is clear to students of education that arithmetic, like every other subject of study in the elementary school, has been largely transformed both as to content and method, in the last few decades. Until recently most of the materials of the elementary curriculum were logically organized and rigidly taught by methods which were quite formal in character. Some subjects were taught largely for their alleged mind-training values. Although arithmetic arose out of a demand to satisfy human needs, many of its processes were taught for their own sake. There was, to be sure, considerable emphasis upon reasoning processes, but memory received the major portion of attention; it was an essential part of practically every class exercise. Ingenious problems, long, involved, and technical, and almost wholly unrelated to the experiences of daily life, could be found on many pages of the arithmetics.

The Factors Responsible for Changes. Three factors have been largely responsible for the changes that have recently occurred in arithmetic. One of these has been the movement for the socialization of the materials; a second has been the result of the scientific study of the problems in arithmetic; and a third has been the movement for the consideration of the processes of arithmetic from the standpoint of their psychological significance. These influences, of course, overlap one another. It is not possible to separate them except for purposes of discussion and emphasis.

The school is a social institution, designed, organized, and maintained by society to save time and labor in training boys and girls in their several duties and responsibilities as citizens. Part of the information and some of the duties and responsibilities of life involve quantitative aspects, which should be included in a course in arithmetic. A careful consideration of these has resulted in a new emphasis being placed upon the character of problem materials found in arithmetics.

It is now maintained that the problem materials must be adapted to the needs and interests of the pupils, instead of attempting to adapt the pupils to them, as was frequently attempted a generation ago. It must be admitted that this has not always resulted in gain to the pupils, because information concerning the needs and interests of the pupils has not always been available. It is clear that the dominant interests of pupils change with age, and vary with all sorts of environmental and social conditions. A pupil in the lower elementary school is, for example, a creature of the present, while a pupil of the grammar school is beginning to project his ideals and to strive to attain some of his life ambitions. The pupil in the lower grades thinks largely in terms of immediate utilities, pleasures, and satisfactions. These motives are still present with the pupils in the grammar grades, but they are less dominant. With maturity, and a broadening of the pupil's educational horizon, his interests may be aroused in ends more or less remote in time.

It is the duty of the teacher to arouse interest in arithmetic, and to utilize the interest thus aroused to secure that careful and systematic study which is a prerequisite of the best educational results. Interest is not something which inheres in a subject; it is not something which exists in the mind of the student remote from and unrelated to the materials of the subject. Persistent application

is the price that must be paid for successful mastery of any subject.

An investigation of the likes and dislikes of children, made at Clark University, shows that a majority of the likes of children for certain subjects correspond to the teacher's like for those same subjects, and that a majority of the dislikes of the children for certain subjects correspond to the teacher's dislikes for those same subjects. The attitudes of teachers, and their emotional accompaniments with reference to the various subjects of study transfer, and find expression in the minds and lives of their students far more quickly than the intellectual processes do. Teachers should remember that interest in a subject lies very near the basis for successful work in that subject.

Problem Materials. Numerous problems relating to the common phases of community life are now being introduced into the arithmetics. It is assumed that problems of this character are more concrete to the pupils as the pupils have some familiarity with their nature and content. Appreciation and interest come partly from understanding, partly from seeing how a thing is to be used, as well as from additional skill in using it. There has been a growing recognition of the fact that there should be a legitimate motive or purpose influencing and stimulating pupils in the solution of problems, and that as many problems as possible should be related to matters that are within the experiences of the pupils.

A problem is not necessarily real to a pupil simply because it appeals to an adult. A problem that appeals strongly to a pupil of the fifth or sixth grade may make but little appeal and may have but little value for a pupil of the second or third grade. Certain number games of the lower grades would be out of place in the upper grades. It is the pupil's intellectual status that must be kept constantly before the makers of textbooks and courses of

study. It is the interest of the pupil, not of the adult, that must receive paramount consideration in the choice of problems.

Myers has pointed out the fact the "children's problems are not merely men's or women's problems cracked up into smaller bits."

It must be remembered that a problem is not of maximum value merely because it is about a factory, a store, a farm, or a bank. Two things to be kept in mind in the selection of problems are:

1. That they should be of a type that is actually met in the common affairs of life.

2. That they should be within the limits of the intelligence and experiences of the pupils.

Most pupils are ready to work with materials that they can understand and whose uses they appreciate. The mere fact that pupils are interested in knowing how to solve a problem is not sufficient. Eagerness to solve problems represents a great gain. Most problems that are drawn from experiences which the children have never yet had, defeat the purpose and violate the principles which we have emphasized in this discussion.

In former years there was a tendency to devote a great deal of attention to the general application of arithmetic. More recently there has been a tendency to deal with problems of a more local character. Between these two extremes, lies the golden mean. It is as unwise to require pupils to solve numerous problems involving applications of which they have no conception as to eliminate all of the interesting types of problems of former years. It is also as unwise to require pupils to solve problems involving applications of which they have no conception as to eliminate or over-emphasize local problems. Those who have adopted the extreme utilitarian view, and who insist that nothing should be taught except that which functions

in the immediate present are not numerous, but their influence is widely felt. In a broad sense any problem in which a pupil is really interested, and which does not give him false ideas, has some practical value for him, because it helps him to develop his understanding of number relations.

Some enthusiasts have recently introduced into the seventh and eighth grades numerous problems that are social in content, but involve the mathematics of the primary grades. The introduction of problems of this type may be of value in classes in elementary sociology and economics, but such problems contribute but little to mathematical insight and skill.

It would be unfair to insist that practical utility is the only criterion by which the worth of a problem may be judged. One of the keenest pleasures which a pupil experiences in the study of arithmetic comes from the ability to overcome the difficulties involved in the solution of problems. The joy of conquest should be a large factor in the intellectual life of a pupil. Too few of our pupils experience the pleasure that comes from the successful mastery of a difficult task. The joy of intellectual effort is too little known by the modern pupil.

The difficulty of securing an adequate supply of practical problems involving the mathematical principles which must be mastered is recognized by many teachers. Great advances have been made in recent years in this respect. Most recent textbooks contain lists of problems dealing with home life, farm life, and the play life of children.

It is a fact, however, that no pupil can acquire facility and accuracy in mathematical operations, and develop sufficiently the power to discriminate and utilize quantitative relationships, if the problems are restricted entirely to the so-called utilitarian types that are immediately within his own experiences. There must be a constant

step-up or advancement or progression in his mathematical experiences. There must be a continual expansion and outward stretch of the mind. A pupil may become greatly interested in the solution of a problem that has no immediate relation to utilitarian ends. A problem that appeals to a pupil just for the joy of the doing should not be left out even though it has no other apparent value.

While it is our purpose to make life as pleasant and the tasks of school as agreeable as possible, we know that we shall meet with many disagreeable experiences both in school and in life outside the school. Some things we must do merely for their own sake; some things we must do because it is our duty to do them; some things we must do as a preparation for later things, even though they are uninteresting. The things that we master now may have a very direct bearing on later work. The thoughtful and skillful teacher welcomes every suggestion that makes any school activity more interesting, and is continually seeking for ways and means to accomplish this end, but no way has as yet been discovered of making all necessary tasks interesting to all of the pupils all of the time.

Emphasis needs to be placed upon having pupils stick to a worth while task until the task is successfully completed, even though the doing of the task may, at the time, not appeal to them as interesting. No good teacher will go to the extreme of giving pupils an uninteresting task to perform merely because it is uninteresting nor will she eliminate everything that does not immediately challenge the interest of the pupil.

The enthusiasm of the teacher will be contagious among the pupils. If the teacher increases her margin of knowledge, and becomes an authority in the field in which she is teaching, she finds that interest begets interest, and the more she puts into her subject the more response she gets from her pupils. Some teachers have neither the

ability nor the energy to enliven a subject, or to awaken interest and enthusiasm by being interested and enthusiastic themselves.

Eliminations and Enrichments. One of the most significant subordinate movements looking to the greater socialization of the materials of arithmetic is the movement for the omission of obsolete topics, as well as the more purely technical ones. There is a tendency for topics to remain in the textbooks long after they have ceased to be useful. This tendency is always present because tradition has an influence in determining practice. Changes which occur in the school studies follow in the wake of changes which occur in the world outside. It is a fact that the reorganization of the materials of instruction is more frequently a result of pressures and sanctions of the world outside than it is the result of leadership within the schools — a leadership which foresees and prophesies the changes which will soon occur in the world, and which prepared the program of instruction in the various studies beforehand, so that pupils may be properly trained to meet these changes.

Not only is it true that obsolete and technical materials are being omitted from the textbooks in arithmetic with greater rapidity than heretofore, but it is equally true and equally significant that the subject is being revitalized by the use of many out-of-school problems. In other words, a more definite attempt is being made on the part of textbook makers to see that students get a usable quantitative point of view concerning the world in which they must live. We must prepare the pupils to meet quantitative needs. A conscious effort is being made to teach pupils those things which they will use later, and at the same time to place them in possession of that information which will enable them to interpret more intelligently, and to appreciate more vividly, the world in which they live.

It would be ridiculous to attempt to present every kind and form of problem that may appear in one's life. A course of study cannot possibly present all of the problems, as well as all of the quantitative aspects or points of view, with which individuals may be concerned in the multitude of professions, trades, and occupations in which they may later variously engage. Consequently, it is necessary to select out of a vast variety of materials and problems, those which will furnish the pupil with the essential constituents, and the necessary quantitative knowledge, that will enable him later to assume his place in the social and industrial world. Solving problems of food, clothing supply, transportation, communication, buying and selling, building, mining, city, county, state and national administration, and the like are properly emphasized.

In recent years problems of the farm have received great emphasis, particularly in the rural communities, and problems of the city, particularly in the urban communities, have received corresponding emphasis.

In 1911, Paul Hanus of Harvard University said, "There is a growing belief that the aim of the work in arithmetic should be limited to accuracy and a reasonable facility in the fundamental operations — addition, subtraction, multiplication, and division of whole numbers and simple fractions; and to simple practical problems involving the operations, together with some instruction in percentage and its simplest applications to interest, trade discount, taxes, and insurance."[1] This statement is as true today as it was in 1911. "The obsolete and the relatively infrequent, the over-complex and wasteful processes of the old arithmetic tend to disappear."[2]

The time saved by the omission of obsolete and technical

[1] Paul H. Hanus, Harvard College, in a Report on the Program of Studies, 1911.

[2] Suzzallo—*The Teaching of Primary Arithmetic.*

topics and the abridgment of others is being utilized in a number of ways. We are beginning to appreciate the extent to which arithmetic may be made to contribute to social insight. By visits to centers of commercial and industrial activity, the pupils are shown how the world uses its mathematics. Instead of devoting all of the time to the solution of the problems about a factory, or a bank, the modern class occasionally visits these institutions to see how the world uses mathematics.

A good modern course in arithmetic includes a consideration of the following topics which have been introduced recently: Graphs, the borrowing of money, the investing of money, modern banking methods, the keeping of simple accounts, the study of tax levies, public expenditures and insurance from a social point of view. Much of the work on some of these topics must be of an informational character, and belongs quite as much to courses in civics and economics as to arithmetic.

The various factors which make for the socialization of arithmetic will, if properly controlled and judiciously used, broaden the pupil's point of view and increase his interest in the subject. Some teachers will carry the idea, no doubt, to absurd extremes, and as a result their pupils may have but little mastery of the essential number facts. Any good tendency may become bad when carried too far.

The mathematical activities of pupils may be classified as follows.

1. Those which are preparatory for later mathematical work.
2. Those needed in carrying on other school work.
3. Those required to meet life's situations outside of school.
4. Those necessary for carrying on a given vocation or profession.

The amount and character of arithmetic necessary for the ordinary duties of life is much less than most people assume. Several investigations have been made in recent years for the purpose of determining what should be omitted from the arithmetic of the elementary school. Only recently has there been corresponding consideration given to the parts of the subject that should receive greater emphasis.

The *American Educational Digest* of July, 1924, stated: "The general conclusions arrived at from studies made to determine what mathematics a person needs to meet life's situations outside of school is that one needs to know, first, how to add, subtract, multiply, and divide whole numbers, common and decimal fractions; second, percentage with applications to the saving and lending of money, mortgages, modern banking methods, building and loan associations, keeping simple accounts, investing money, bonds as investments, real estate as investments, the marks of a good investment, taxes, levies, public expenditures, profits in different lines of business, and life insurance as protection and investment; third, practical measurements, such as liquids, solids, time, and others; fourth, graphical representation of data including the use of frequent distributions and central tendencies; fifth, simple equations."

The Socialization of Arithmetic. The changes that have occurred in the organization of materials for the teaching of arithmetic is affecting the socialization of the subject. These changes have expressed themselves in two directions. The first relates to the changes that have occurred in the organization of the drill phases of the subject and the second to the content phases of the subject.

Until recently arithmetic was regarded essentially as a tool subject. Emphasis was placed upon the acquisition of skill, and there was an enormous amount of formal drill to enable the pupil to fix in mind the fundamental number

facts. All sorts of ingenious exercises were invented to develop the desired skill. Facility and accuracy in computation were constantly emphasized. The tables and the various number combinations were taught often without relation to problems. One multiplication table after another was taught until the children were able to recite them without making any mistakes. Practically all of the number facts were taught as if they were of equal difficulty.

Procedure with reference to the acquisition of skill in computation has been almost completely transformed within recent years. Investigations have shown that the various number combinations are not of equal difficulty, that they should not receive equal emphasis and that their distribution throughout the course of study is a matter of importance. No longer are they introduced into the best textbooks in a more or less miscellaneous and random fashion. Instead they are being distributed in accordance with a scientifically organized arrangement somewhat in the order of their relative difficulty. In most cases the use of the various combinations is revealed to the pupils in real situations and problems.

After the pupils have been made to see and understand and appreciate the importance of increased facility in the handling of these number combinations, the combinations are then selected out of their natural setting and given a certain amount of formal attention. There is no reason to suspect that the children of this generation handle the number combinations poorly. There is evidence to prove that the children of today actually do better with them than did the children of two or three generations ago.

That the materials of arithmetic are being organized to increase their socializing influence is further evidenced by the fact that plays and games using number relations appear with greater frequency in the lower grades than

formerly, and that, in the upper grades, the fundamental principles are reviewed, emphasized, and made more meaningful by being used in the solution of problems centering around common practical projects. It is not uncommon to find fifty or a hundred problems related to the activities of the community, some profession or trade, or some form of public service. These problems are so arranged as to call for a review of the essential number facts which the students have mastered.

Much of the poor work which characterized arithmetic heretofore has been due to indifferent and poorly established habits of reading the problems. With an improvement in reading methods there has been a corresponding improvement in most of the other subjects of study, including arithmetic. Pupils need a great deal of instruction in interpreting problems. This is a phase of the work that has received too little emphasis in most schools. It is wise to require pupils frequently to analyze the data given in a problem in order to determine the processes to be used, writing down in one column the analysis in systematic form and in another column doing the necessary computing, using all of the short cuts possible. In the one column emphasis should be placed upon accuracy of the processes; in the other upon accuracy of computation.

Originality and Initiative. Not only is there a tendency to master more completely the quantitative side of life, but there is a decided tendency to foster a spirit of inquiry and of originality in interpreting number relations. Instead of requiring a pupil to solve a problem according to some type form, the teacher should encourage his originality by permitting flexibility in analysis. The pupil should be encouraged to choose the method that seems best to him, and then he should be asked to justify his choice.

A pupil tends to associate a particular process with a specific form of statement; he is often unable to solve a

problem when the phraseology is changed even slightly. Marked variations in the difficulty of a problem may be caused by differences in phraseology. By avoiding uniformity in phraseology, a pupil may be forced to rely more upon his own resources in the solution of problems. A pupil should be encouraged to see as quickly as possible the relation between what is given and what is required, and to choose the appropriate process.

Difficulty in a given problem frequently lies in the fact that a pupil is not able to break the problem up into its various steps and to apprehend the order of the steps. Unless he has ability to analyze problems, his knowledge of arithmetic will be more or less mechanical. The pupil who continually depends upon rules or type problems has attained little mastery of the subject. He should be encouraged to devise solutions of his own. The more insight he has into the relationship involved in a given problem the freer he will be from mechanical procedure.

The Unity of Arithmetic. The tendency to develop arithmetic as a whole, rather than as a series of more or less unrelated topics, has been gaining ground. More emphasis is being placed on the mastery of principles, and less on the mastery of rules and definitions. There is in the best textbooks in arithmetic today a minimum of theory and a maximum of practice. The pupil who fails to grasp the few underlying principles of arithmetic has seen but little of the simplicity and beauty of the subject.

The number of distinct mathematical principles involved in arithmetic is surprisingly small, but unless a pupil grasps these principles the whole subject lacks organic unity in his mind. It is important to analyze and to classify topics, to be able to distinguish points of similarity and of difference; to develop the power to see relations between topics that are based upon the same underlying principles.

A pupil who does not see the unity in arithmetic feels that he is dealing with something entirely new from the mathematical point of view, every time he takes up a new topic, whereas he may be dealing with a new phase of a topic long familiar to him. Real knowledge implies organization and organization in turn implies more or less unity. Spencer says, "When a man's knowledge is not in order, the more of it he has the greater will be his confusion of thought. When the facts are not organized into faculty, the greater the mass of them, the more will the mind struggle under its burden, hampered instead of helped by its acquisition."

There is a tendency today, not only to unify the subject of arithmetic, but to introduce more or less unity in the assignment of problems for a lesson or group of lessons. The custom has been to group problems more or less in a series involving the same processes but different in other respects. The present tendency is "to attain a more approximate unity within the subject matter of the problems themselves."

Some Influences of Scientific Studies. As was stated in the opening paragraph of this chapter, the movement for the socialization of arithmetic is not the only significant present tendency in this field. Attention has been directed to the influence of scientific studies in modifying our conceptions of the work in this field. This movement has, of course, touched all of the elementary subjects, but none more powerfully than arithmetic. Perhaps its most significant tendency has been the effort to establish the various methods of presenting arithmetic upon a scientific rather than upon an empirical basis.

There have been numerous attempts to determine by means of controlled experiments the relative importance of various methods of teaching. There are those who have been loathe to accept the results of these investigations;

partly because of a reluctance to modify practice, and partly because they assert that some of the alleged scientific investigations have been unscientific. Not every suggestion for reform is in the right direction; it is as necessary to reject new errors as it is to eliminate old ones. We must not lose our bearings in the midst of the educational unrest of the day. We must examine with great care that which the pedagogic alchemist would have us try. We must follow the conclusions of our best thought and not be bound by the prejudices of years of practice.

The basis for any successful system of pedogogy must be laid in the experiences of preceding generations, and in accordance with scientific methods of procedure. It is necessary to master the pedogogy of the mathematics of the past, its aims, its methods, and devices in order to estimate the advantage of previous attainments, and in order to make a wise selection among the many suggestions and schemes for improvement. There must be a careful weighing and adjustment. Through a knowledge of the history of arithmetic we gain an ability to deal with modern suggestions for improvement which we are called upon to judge. We also avoid the ridiculous situation of advocating, as new, a method or device proposed and tried out centuries ago.

The teacher should be familiar with the methods of modern scientific experiments and conversant with their results. More of our teaching is now based upon the results of such investigations than ever before. There has been far too much passive acceptance of general practice in education. Not every teacher has the ability or training that justifies him in attempting to plan and to conduct an experiment to determine the relative values of different methods of procedure. Numerous factors may enter into an investigation, and, unless these are given proper weight, the experiment may be of no value. Most teachers, how-

ever, can co-operate with an investigator of ability, and thus aid him in increasing the amount of knowledge based on controlled experiment.

Tests and Scales. The scientific movement in education has expressed itself in arithmetic in the form of tests and scales which were used for some time largely as survey instruments. Tests were given and results carefully compiled. The conditions that were discovered were often discouraging and showed there was a need for a careful study of the whole arithmetic situation. These tests were often not understood by teachers and they made little use of the results. Happily this condition no longer exists. Most teachers have discovered that these tests are of real value to them. They can quickly discover the general average of a class as compared with the country as a whole or as compared with other classes in the same city. They can locate pupils who are in need of special attention.

There is now an abundance of literature available which describes the results of studies that have been made to improve instruction and learning in arithmetic. By means of these tests, repeated at the end of the year, teachers can determine the amount of progress by the pupils in their classes. Certain tests are also of value because pupils are enabled to discover their own weaknesses in arithmetic. This is an inducement to better work since the children know their own difficulties and their co-operation is more easily secured in attempting to raise their abilities to a higher level of efficiency.

Textbooks. Another influence which the scientific movement has had, as we have already indicated, has manifested itself in a changed character of the textbook. Some years ago, when the faculty psychology was in vogue, the logical arrangement of topics was emphasized. The disciplinary side of arithmetic was emphasized more than the practical side. Gradually the demands for the elimination of the

obsolete and the impractical, and for the reorganization of the remaining materials became more insistent.

The importance of giving attention to the proper organization of textbooks can scarcely be over-emphasized. Thousands of teachers are largely dependent upon the textbooks for their methods, organization, materials, and for the sequence of topics. Originality and initiative are probably as rare among teachers as among individuals in other professions.

Many of the older texts were followed slavishly; a few of the texts even indicated the questions that the teacher was to ask, and specified the pages upon which the answers were to be found.

There is today a growing recognition of the fact that no textbook can adequately meet the needs of any class at every instant. It is the duty of the teacher to introduce numerous exercises not in the text, to give many of the problems a local setting, and to omit such topics and problems as are not adapted to the needs and abilities of the pupils. It was formerly regarded as a confession of weakness for a teacher to omit any topic or problem from the text. We are beginning to recognize that judicious elimination and supplementing of the materials of the textbook is an indication of strength.

Many teachers are prone to view with suspicion any method of procedure that a pupil ventures to introduce unless it corresponds with the methods of the text. Such an attitude tends to repress, rather than to encourage, that spirit of inquiry and of originality which arithmetic should seek to cultivate. The use of a textbook is, in most cases, economical both of time and energy. The textbook should serve as a basis for work. A good teacher will get good results in spite of a poor textbook, but a well written text increases the efficiency of a teacher. A poor teacher will get poor results with any textbook.

The Course of Study in Arithmetic. The necessity of revising the curriculum, as well as the textbook in arithmetic, is recognized as a nation-wide problem. In the spring of 1924 the president of the National Education Association appointed a national committee on curriculum revision in arithmetic. This committee hopes to secure the co-operation of educators in all parts of the country to aid in the work. The results of their investigations will undoubtedly greatly influence the work in arithmetic during the next few years. One phase of this committee's work is concerned with the determination on a much larger scale than has been possible before the occupational and social significance and needs of arithmetic. Such studies as those of Wilson and Woody, to which attention was called in the chapter on *Scientific Studies in Arithmetic*, will be greatly extended. Similar investigations will be made in the fields of drill and remedial work. A definite effort will be made to eliminate much of the waste in classroom procedure.

The tendency which has manifested itself in various quarters to break away from the text and to prepare courses of study in arithmetic of a purely local nature is not to be commended. Materials of this character may be used for supplementary purposes to advantage, but a curriculum that is organized entirely around local projects lacks that universality which is so essential for wider use. A good course of study involves a large number of factors of general significance and application.

The Psychology of Arithmetic. Following close in the wake of the movement for the scientific study of arithmetic, there is an attempt to study the psychology of the subject. Extensive investigations are being made to discover the best methods of teaching children the various processes in arithmetic. Each process is being carefully analyzed and the difficulties which children encounter are being studied. Much poor teaching resulted formerly from

the assumption that the various processes and operations should be taught in a certain logical sequence. Now increased emphasis and attention is paid to the psychological rather than to the logical factors involved in the mastery of number facts and relations.

Pupils who are having unusual difficulties in arithmetic are having their work carefully studied and an analysis made of specific causes. Where this has been done, remedial work is being prescribed to aid in overcoming these weaknesses. These investigations make it clear that many faulty methods of work are acquired by pupils under the usual plan of mass instruction.

Standards are being developed by means of which a teacher can evaluate the problems that are used in the class work in arithmetic. Much of the unnecessary and useless content of the subject is being eliminated.

It has been shown that the traditional notion that there are only forty-five fundamental combinations in addition is an erroneous one. The number of combinations that must be learned by a pupil who is to master all of the fundamental operations is more than thirteen hundred.

Arithmetic textbooks are being analyzed to discover the extent to which they make use of the results of scientific research. Many of them have been found to be faulty in many respects.

Studies are being made to determine the effectiveness of the various methods employed in teaching. Methods are checked in terms of the results secured. The so-called problem method, which has been advocated and used rather widely in most of the subjects of study, is being scrutinized, checked, and tested in connection with practically every operation in arithmetic. It has been found that many of the problems are as remote from the experiences and abilities of the children as were the logical operations in the traditionally organized text.

It must be clear that changes that have occurred in this field of arithmetic in the last ten or twenty years have been remarkable, but probably not less so than those which will occur within the next decade. Serious minded students of education in many countries are at work upon the problems in this field. A readjustment and a re-evaluation of the materials is going on constantly. Methods of teaching are being scrutinized and examined and tested with the aid of instruments unknown twenty years ago. More training, greater skill and ability are required on the part of the teachers today than was required ten years ago. The contents of the textbooks, and of the courses of study, will be socialized more and more and the socialization of the subject will necessitate new readjustments from time to time. Science will refine its instruments of investigation, and psychology will continue its interpretation of the several processes of thought which are involved.

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SOME SUGGESTIONS FOR THE SUPERVISOR

The following list is not complete, but it indicates the opinions of a large number of experienced superintendents, principals, and supervisors, with respect to the information which a supervisor of arithmetic should have. In addition to knowing the answers to the questions, the supervisor ought to be thoroughly competent to show a teacher just how to overcome a weakness in method or how to enrich the arithmetic work as a whole. In other words, a supervisor must, first of all, be a skillful teacher. In addition to this skill, a supervisor must have a broad background and be thoroughly informed on the work that is being done in his field.

What a Supervisor Should Know About Arithmetic

1. *Status of arithmetic in surveys.*
   (a) What has been the history of arithmetic in our schools?
   (b) What did Rice, Stone, Courtis, and others discover concerning the status of arithmetic?
   (c) What has been the status of arithmetic as revealed in the city and state surveys that have been made?

2. *Content of arithmetic in the light of scientific investigations.*
   (a) What abilities may be said to compose the complex of abilities called "arithmetic ability"?
   (b) How can arithmetic material be best graded?
   (c) What is meant by the fundamental combinations in addition, subtraction, multiplication and division? What are they? How many are there of each?

(d) What studies have been made to determine the arithmetic needs in various occupations?
(e) Should the content of a course of study be based on adult needs or child needs?
(f) What are worth while projects or group activities that have been tried out by teachers and found to contribute to the growth of the child in arithmetic ability?
(g) What are the objectives of work in arithmetic in the schools as a whole? For each grade?
(h) What type of faulty thought problems are sometimes used by teachers?
(i) What are some of the best courses of study in arithmetic?
(j) How do textbooks for the grades make use of these investigations?

3. *Methods in the light of scientific investigations.*

(a) What reference books on methods for teachers of arithmetic are available?
(b) What games and devices are useful in teaching the child the arithmetic concepts in the primary grades?
(c) What drill devices, standardized and unstandardized, are available, by means of which each child of a group may be given the drill in the particular difficulty he has?
(d) What mistakes are made by teachers in handling the standard practice materials?
(e) What devices are useful in helping a teacher diagnose the difficulties a pupil may have in certain operations? What are the commonest types of difficulties in fundamentals and problem work? How are difficulties diagnosed?

(f) What are the best types of remedial work for each difficulty?
(g) By what standards should drills be regulated?
(h) Which of the fundamental operations are of the greatest difficulty? Least?
(i) What is the value of the intensive campaign method of instruction?
(j) How can arithmetic work be motivated?

4. *Tests and measures of achievement in arithmetic.*

(a) What tests should a teacher use to determine the general state of the arithmetic abilities of the pupils in his class?
(b) What is the best way to present graphically the results of a test? How are the tests given, scored, and tabulated?
(c) How can the best use be made of the results of a test?
(d) What tests are available for survey purposes? In
(1) Fundamentals and fractions?
(2) Thought problems?

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